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SUPERDIFFUSIVE TRANSPORT DUE TO RANDOM VELOCITY FIELDS ☆

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The motion of a random walk in a medium containing random, but spatially correlated velocity fields is discussed. This type of disorder generally leads to *superdiffusive* behavior in which the mean-square displacement, $\langle x^2(t) \rangle$, grows faster than linearly with time. For a two-dimensional medium with layers of random velocities in the x-direction, $\langle x^2(t) \rangle$ is found to increase as $t^{2\nu}$ with $2\nu = 3/2$. The probability distribution of displacements appears to fit the form $P(x, t) \sim t^{-3/4} \exp[-(x/t^{3/4})^{\delta}]$, with $\delta \lesssim 1.7$. However, a Lifshitz argument for the tail of the distribution suggests that $\delta \leq 4/3$. This discrepancy is yet to be fully resolved. We also discuss some intriguing properties in a model with isotropic velocity fields. In two dimensions, we find $\nu = 4/3$, while $\delta = 3$. These values obey the general scaling relation $\delta = (1-\nu)^{-1}$.

1. Introduction

Stochastic transport in spatially heterogeneous media often leads to anomalous, or subdiffusive, behavior in which the mean-square displacement, $\langle x^2(t) \rangle$, grows more slowly than linearly with time. This has been a topic of enormous interest in the recent past (see e.g. refs. [1–3]). An essential mechanism for this anomalous behavior is the presence of disorder on all length scales. As a function of time, the random walk explores regions of progressively higher "resistance" to transport. This leads to a diffusion coefficient which is a decreasing function of length scale, or time. The vanishing of the diffusion coefficient at large scales can be viewed as the mechanism that leads to anomalous diffusion. Corresponding to this anomalously slow transport, the probability distribution is generally non-Gaussian in nature.

In this article, I discuss a very simple mechanism that gives rise to the complementary situation of *superdiffusive* behavior, in which $\langle x^2(t) \rangle$ grows faster than linearly in time. Owing to the relative simplicity of the models for which this behavior can be realized, this general phenomenon should prove to be a productive area for future work. The models that I will discuss are based on the coupling between diffusion and random velocity fields. The physical motivation for this class of models arises in attempting to describe ground water transport in macroscopically heterogeneous rocks [4]. Consider, e.g., a two-dimensional stratified medium consisting of strips which are infinitely long in the x direction and of random widths in the y direction. Each layer is homogeneous but distinct, so that transport properties may vary from layer to layer. This might describe a two-dimensional section of a typical sedimentary rock.

Suppose that fluid is flowing in the x direction (along the strata) and that diffusive mixing takes place between layers. Owing to the differences in each layer, the fluid velocity correspondingly varies from layer to layer. In a center-of-mass frame of reference, then, there are random velocities in the x direction, and pure diffusion in the y direction. If a random walk is released into the flow, then these two driving forces lead to an anomalously

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large, superdiffusive spread of the probability distribution. Some of the intriguing features associated with this class of transport processes are outlined below.

2. Unidirectional random velocities

For the case of random layers of differing velocities, consider the following simple lattice model on the square lattice. Each line in the x direction is randomly assigned a velocity $\pm$. At each vertex of the lattice, this means that a random walk moves either in the $+y$ or $-y$ direction with probability 1/4, and moves along the direction of the pre-assigned velocity in the x direction with probability 1/2. This corresponds to an infinite velocity bias in the longitudinal direction. For this system, the mean-square displacement of a random walk at time t increases as $t^{3/2}$, as Matheron and de Marsily [4] were apparently the first to recognize. This remarkable result arises because in a time t a random walker explores of the order of $D_\perp t$ horizontal layers, where $D_\perp$ denotes the (microscopic) diffusion coefficient in the transverse direction. Although the longitudinal velocity averaged over an infinite number of layers is zero, the average over a finite number of layers will generally be fluctuating and a vanishing function of the number of layers that the random walk visits. This non-vanishing bias underlies superdiffusive transport.

The average velocity at time t is given by

$$\langle v \rangle_t = (D_\perp t)^{-1/2} \sum_{i=1}^{(D_\perp t)^{1/2}} v_i \sim (D_\perp t)^{-1/4}. \tag{1}$$

The average has been taken only over the $D_\perp^{1/2}$ layers that a typical random walk visits. Correspondingly, the rms longitudinal displacement at time t, $x_{\rm rms}(t) \equiv \langle x^2(t) \rangle^{1/2}$, may be estimated as

$$x_{\rm rms}(t) \sim \langle v \rangle_t t \sim D_\perp^{-1/4} t^{3/4}. \tag{2}$$

In addition to investigating the moments of the probability distribution, it is important to study the probability distribution itself [5]. Our preliminary work suggests that the tail of this distribution function is governed by a Lifshitz-type singularity whose manifestations are extremely difficult to observe numerically, while the peak of the distribution shows markedly different behavior.

As a natural first hypothesis, the probability distribution for the longitudinal displacement, averaged over all configurations of random velocities, is taken to have the form

$$\langle P(x,t) \rangle \propto t^{-\nu} \exp[-(x/t^{\nu})^{\delta}], \tag{3}$$

with $\nu = 3/4$. Our goal is to find the shape exponent δ. A Gaussian distribution corresponds to $\delta = 2$, while for many stochastic walk models, an argument by Fisher [6,7] generally gives $\delta = (1-\nu)^{-1}$. For random velocity layers, this would yield $\delta = 4$. In order to test the hypothesis of eq. (3), we have performed three independent calculational schemes, all of which yield the same result. In these methods, we first compute the dimensionless moment ratios,

$$m_{2k}(t) \equiv \langle x^{2k}(t) \rangle / \langle x^2(t) \rangle^k. \tag{4}$$

The m_{2k} are found to approach constants as $t \to \infty$ whose values depend on δ. By attempting to match these moments to those that arise by directly computing the moments from eq. (3), we infer a value $\delta \lesssim 1.7$.

One way to compute the moments is by a direct evaluation of the stochastic integral over the differing paths that the random walk takes in the transverse direction. By the definition of the displacement, we have

$$\langle\langle x(t)\rangle_w\rangle_c = \int_0^t dt' \, \langle\langle u(y(t'))\rangle_w\rangle_c, \tag{5}$$

where we have explicitly denoted that the average is first performed over all random walk trajectories, and then over all configurations of random velocities. The moments of the displacement can therefore be written as

$$\begin{aligned}\langle\langle x^n(t)\rangle_w\rangle_c &= \int_0^t dt_1 \ldots \int_0^t dt_n \, \langle\langle u(y(t_1))\ldots u(y(t_n))\rangle_w\rangle_c \\ &= n! \int_0^t dt_1 \int_0^{t_1} dt_w \ldots \int_0^{t_{n-1}} dt_n \, \langle\langle u(y(t_1))\ldots u(y(t_n))\rangle_w\rangle_c .\end{aligned} \tag{6}$$

The time-ordered product of the velocity correlation function is

$$\begin{aligned}&\langle\langle u(y(t_1))\ldots u(y(t_n))\rangle_w\rangle_c \\ &= \int_{-\infty}^{+\infty} dy_1\, dy_2 \ldots dy_n \, \langle u(y_1)\ldots u(y_n)\rangle_c \, p(y_n, t_n)\, p(y_{n-1}-y_n, t_{n-1}-t_n)\ldots p(y_1-y_2, t_1-t_2),\end{aligned} \tag{7}$$

where

$$p(x,t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt) \tag{8}$$

is the Gaussian probability distribution for the motion in the transverse direction. The evaluation of the integral in eq. (6) is straightforward, but quite tedious. Up to the sixth order (only even powers of velocities within a given layer give non-zero contributions), we find the moment ratios $m_4 \simeq 3.3$ and $m_6 \simeq 19.1$. These are reasonably consistent with the value $\delta \simeq 1.7$ quoted above. These results are also corroborated by direct Monte Carlo simulations, and by a direct enumeration of the occupancy correlation function of the one-dimensional random walk problem in the transverse direction.

On the other hand, a Lifshitz-type argument based on considering the walks which contribute to the extreme large-distance tail of the distribution suggests that $\delta \leq 4/3$. Consider the average probability of a "stretched out" walk, namely, $\langle P(x \sim t, t)\rangle$. According to eq. (3), this should vary as $\exp(-t^{\delta/4})$. On the other hand, the probability of a walk being stretched out can be bounded from below by the probability of remaining confined to one "avenue" in which the velocity bias is in the same direction. In direct analogy with the survival probability of a one-dimensional random walk in the presence of randomly distributed traps [8], the confining probability, averaged over all configurations of the environment, varies as $\exp(-at^{1/3})$. This implies the exponent inequality, $\delta \leq 4/3$. As suggested by our various calculations, the asymptotic behavior is likely to be masked by very slow crossover effects.

3. Generalizations to higher dimensions and to isotropic velocities

Consider the quasi-two-dimensional problem in which the y–z plane consists of random "filaments" in the $\pm x$ direction. (This can be generalized to d'-dimensional strata in a d-dimensional system.) For this system, the arguments leading to eqs. (1) and (2) can again be invoked. Now a random walk visits of the order of $t/\ln t$ different filaments in a time t. Thus $\langle v\rangle_t \sim (\ln t/t)^{1/2}$. Consequently, the rms displacement should vary as $(t \ln t)^{1/2}$. The logarithmic correction to otherwise diffusive behavior suggests that the upper critical dimension for this system is 3.

The behavior of the probability distribution of the displacements also appears to be quite interesting. On the basis of the diffusive behavior of the $x_{\rm rms}$, one might expect that $\langle P(x,t)\rangle$ would be much closer to a Gaussian than in the two-dimensional layered model. On the other hand, a strict (and naive) application of the Lifshitz-tail argument suggests that the exponent inequality $\delta \leq 1$, even further from Gaussian behavior than in two dimensions. The mechanism by which the Lifshitz-tail argument breaks down appears to be quite subtle and interesting. No reliable numerical results for three dimensions are as yet available. The resolution of these conflicting results should prove to be quite interesting.

A second general class of models is the motion of a random walk in the presence of random, but *isotropic* random velocities. For example, consider a random walk on a random "Manhattan" grid, in which the directionality along any Avenue or Street is fixed along its entire length, but whose orientation is random. For this system, we generalize the arguments of eqs. (1) and (2) by decomposing the motion into transverse and longitudinal components, even though the motion is isotropic. Assuming that $x_{\rm rms} \sim t^{\nu}$, then from eq. (1), the mean velocity, averaged over these t^{ν} layers, vanishes as $t^{-\nu/2}$. From eq. (2), one then concludes that $x_{\rm rms} \sim t^{1-\nu/2}$. Since the motion is isotropic, however, we must have $\nu = 1-\nu/2$, or $\nu = 2/3$. Generalizing these arguments to higher dimensions suggests that $\nu = 2/(d+1)$ for spatial dimension $d \leq d_{\rm c} = 3$.

For the probability distribution of displacements in two dimensions, relatively modest simulations indicate that eq. (3) holds with the exponents $\nu = 2/3$ and with $\delta = 3$. Interestingly, these values are in accord with the general Fisher argument between the shape and size exponent, $\delta = (1-\nu)^{-1}$. It is slightly mysterious that the Fisher argument appears to work for isotropic random velocities but fails for layered random velocities.

4. Summary and discussion

We have discussed some intriguing aspects of superdiffusive transport that occur in media which possess random, but correlated velocity fields. By elementary arguments, one can understand how superdiffusive transport arises from the interplay between diffusion and the geometry of the random velocity fields. For the layered system, it is possible to develop formal calculational schemes to obtain the positive integer moments of the probability distribution. However, the utility of these approaches appears to be limited by the existence of Lifshitz-type singularities which control the asymptotic behavior of the distribution. For an isotropic two-dimensional "Manhattan" system, the probability distribution appears to exhibit conventional scaling in which $\delta = (1-\nu)^{-1}$.

While the work thus far has focused on the spatial moments of the probability distribution, it should also prove fruitful to study the first passage probabilities in superdiffusive transport processes. The mechanisms that yield the basic features of the distribution of first passage times between an input and absorber may well be very different from those that describe the spatial moments of the distribution. In short, there are still a wide variety of fundamental, puzzling questions for which satisfactory, first-principles explanations are still lacking.

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