Majority versus minority dynamics: Phase transition in an interacting two-state spin system

M. Mobilia and S. Redner*
Center for BioDynamics, Center for Polymer Studies & Department of Physics, Boston University, Boston, Massachusetts 02215, USA
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We introduce a simple model of opinion dynamics in which binary-state agents evolve due to the influence of agents in a local neighborhood. In a single update step, a fixed-size group is defined and all agents in the group adopt the state of the local majority with probability p or that of the local minority with probability $1-p$. For group size $G=3$, there is a phase transition at $p_c=2/3$ in all spatial dimensions. For $p>p_c$, the global majority quickly predominates, while for $p<p_c$, the system is driven to a mixed state in which the densities of agents in each state are equal. For $p=p_c$, the average magnetization (the difference in the density of agents in the two states) is conserved and the system obeys classical voter model dynamics. In one dimension and within a Kirkwood decoupling scheme, the final magnetization in a finite-length system has a nontrivial dependence on the initial magnetization for all $p\neq p_c$, in agreement with numerical results. At p_c, the exact two-spin correlation functions decay algebraically toward the value 1 and the system coarsens as in the classical voter model.

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I. INTRODUCTION

In this paper, we investigate the properties of a simple model of opinion formation. The model consists of N agents, each of which can assume one of two opinion states of $+1$ or -1. These agents evolve according to the following rules (Fig. 1).

(1) Pick a group of G agents (spins) from the system, with G an odd number. This group could be any G spins in the mean-field limit, or it could be a randomly chosen contiguous cluster of spins in finite-dimensional systems.

(2) With probability p, the spins in the group all adopt the state of the local majority. With probability $1-p$, the spins all adopt the state of the local minority.

(3) Repeat the group selection and attendant spin update until the system necessarily reaches a final state of consensus.

We term this process the *majority-minority* (MM) *model*, in keeping with the feature that evolution can be controlled either by the local majority or the local minority. The MM model represents a natural outgrowth of recent analytical work on the *majority rule* model of opinion formation [1], which, in turn, represents a particular limit of a class of models introduced by Galam [2]. In majority rule, the opinion evolution of any group is controlled only by the local majority within that group. Thus majority rule corresponds to the $p=1$ limit of the present MM model.

A basic motivation for this type of modeling is to incorporate, within a minimalist description, some realistic aspects of the manner in which members of an interactive population form consensus on some issue. In this spirit, the MM model allows for the possibility that a forceful and/or charismatic minority can sometimes dominate the opinion of a group, an experience that many of us have had in our everyday lives. The limit where p is close to 1 is probably closer to socially realistic situations. Part of our interest in considering the case of general p is to understand the change in dynamics as a function of p and the kinetic phase transition that occurs at p_c.

We shall see that the interplay between minority and majority rules leads to three distinct kinetic phases in which the approach to ultimate consensus is governed by different mechanisms. As in the earlier work on majority rule [1], we seek to understand the long-time opinion evolution. We will be primarily concerned with determining the probability of reaching a given final state (the exit probability) as a function of p and the initial densities of each opinion state.

To provide perspective for this paper, we briefly review related work on opinion dynamics models. Perhaps the simplest such example in this spirit is the classical voter model [3]. Here a two-state spin is selected at random and it adopts the opinion of a randomly chosen neighbor. This step is repeated until a finite system necessarily reaches consensus. One can think of each spin as an agent with zero self-confidence who merely adopts the state of one of its neighbors.

An attractive feature of the voter model (in contrast to the familiar Ising model with Glauber kinetics [4]) is that it is exactly soluble in all spatial dimensions. For a finite system of N spins in d dimensions, the time to reach consensus scales as N for $d>2$, as $N\ln N$ for $d=2$ (the critical dimension of the voter model), and as N^2 in $d=1$ [3,5,6]. In d

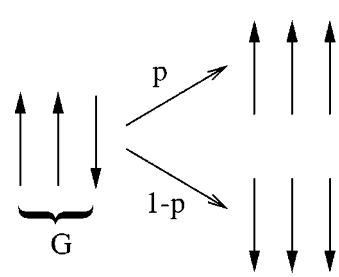

FIG. 1. Evolution of a group of $G=3$ spins according to MM dynamics. Majority rule applies with probability p and minority rule applies with probability $1-p$.

*Electronic address: mmobilia,redner@bu.edu

$=1$ and 2, an infinite system coarsens so that consensus emerges on progressively larger length scales, while for $d>2$, an infinite system approaches a steady state of mixed opinions. Because the average magnetization is conserved [3], the probability that the system eventually ends with all plus spins equals the initial density of plus spins in all spatial dimensions.

From a more practically minded viewpoint, there has been a recent upsurge of interest in kinetic spin-based statistical physics models that attempt to incorporate some realistic sociological features. One such example is Galam's rumor formation model [2,7], in which a population is partitioned into variable-sized groups, and in each update step the spins in each group may adopt the majority state or the minority state of the group depending on additional interactions. Our majority model represents a special case in which only a single group of fixed size G is updated at each step. Another prominent example is the Sznajd model, where spins evolve only when local regions of consensus exist [8]. In the basic version of the model, when two neighboring spins are in the same state, this local consensus persuades a neighboring spin to join in. Such a rule naturally leads to eventual global consensus except in the anomalous case of an antiferromagnetic initial state. The generic questions posed above about opinion evolution in the MM model are also of basic interest in the Sznajd model [9] and considerable work has recently appeared to quantify its basic properties [9–13]. There is also a wide variety of kinetic spin models of social interactions that incorporate, for example, multiple traits [14], incompatibility [15,16], and other relevant features [17].

An inportant feature of our MM model is that the competition between majority and minority rules leads to a kinetic phase transition in all spatial dimensions d at a critical value of $p_c=2/3$ for group size $G=3$. The existence of such a transition can be easily understood by considering the average change of the magnetization in a single update step. A group undergoing an update must consist of two spins of one sign and a single spin of the opposite sign. According to Fig. 1, the magnetization change in such a group is proportional to $2p-4(1-p)$, which is zero when $p=p_c=2/3$. For $p>p_c$ and for all $d\geqslant 2$, the system quickly evolves toward global consensus where the magnetization equals ± 1 [18]. For $p=p_c$, the average magnetization is conserved, as in the voter model. Consensus is again always reached, but the time until consensus grows as a power law in time. For $p<p_c$, the system is driven toward a state with equal densities for the two species of agents. Since consensus is still the only absorbing state of the dynamics, consensus is eventually reached in a finite system, but the time needed grows exponentially with the system size. It bears emphasizing that for all p and for all d, a finite-size system necessarily reaches consensus in the MM model. There are no metastable states that prevent the attainment of ultimate consensus as in the related majority vote process [3] or in the zero-temperature Ising Model with Glauber kinetics [19].

The MM model exhibits special behavior in one dimension in which the magnetization quickly approaches a static value that depends only on the initial magnetization. If one focuses on the interfaces between domains of agents in the same state, these domain wall particles undergo the diffusive annihilation reaction $A+A\to 0$, but with constraints in the motion of domain walls, when they are nearby, that reflect the constraints of the MM dynamical rules. Our understanding of this intriguing aspect of the problem is still incomplete.

In Sec. II, we investigate the exit probability and exit times in the mean-field limit of the MM model. We then turn to the case of one dimension in Sec. III. We first write the master equation for the configurational probability distribution, following the original Glauber formalism. We apply a Kirkwood decoupling scheme [20] for correlation functions to compute the final magnetization as a function of the initial magnetization. Finally, we show that in the exactly solvable case of $p=p_c=\frac{2}{3}$, the two-spin correlation function $c_r(t)\equiv\langle S_i(t)S_{i+r}(t)\rangle$ approaches one as $t^{-1/2}$ for all r. Thus the system exhibits diffusive coarsening, as in the traditional voter model. We give a summary and discussion in Sec. IV. Calculational details are given in the appendices.

II. THE MEAN-FIELD LIMIT

A. Exit probability

Following the approach developed in Ref. [1], we first study the exit probability E_n, namely, the probability that a system that initially contains n up spins in a system of N total spins ends with all spins up. This exit probability obeys a simple recursion relation in which E_n can be expressed in terms of the exit probabilities after one step of the MM process [21].

To construct this recursion relation, we note that

$$p_n\equiv 3p\binom{N-3}{n-2}\Big/\binom{N}{n} \quad \text{and} \quad q_n\equiv 3p\binom{N-3}{n-1}\Big/\binom{N}{n}$$

are the respective probabilities that a group of three spins contains 2 plus and 1 minus spins or contains 1 plus and 2 minus spins, and that the majority rule is applied to the group. Thus p_n is the probability that there is a change $n\to n+1$ and q_n is the probability that there is a change $n\to n-1$ in a single step of the MM process. Similarly

$$\overline{p}_n\equiv 3q\binom{N-3}{n-1}\Big/\binom{N}{n} \quad \text{and} \quad \overline{q}_n\equiv 3q\binom{N-3}{n-2}\Big/\binom{N}{n},$$

with $q=1-p$, are the respective probabilities for n to change by ± 2 steps due to minority rule being applied to the group. The master equation for the exit probability is [21]

$$E_n=\overline{p}_nE_{n+2}+p_nE_{n+1}+q_nE_{n-1}+\overline{q}_nE_{n-2}. \tag{1}$$

While the exact solution to this discrete recursion relation was given in Ref. [1] (for $p=1$), it is much simpler to consider the continuum limit of $n,N\to\infty$ with $x=n/N$ finite. In this limit, the hopping probabilities reduce to

$$p_n=3px^2(1-x),\quad q_n=3px(1-x)^2,$$

$$\overline{p}_n=3qx(1-x)^2,\quad \overline{q}_n=3qx^2(1-x),$$

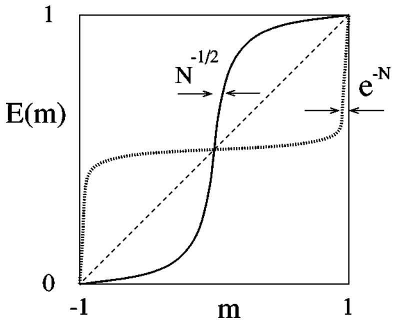

FIG. 2. Sketch of the exit probability $E(m)$ that a finite system with initial magnetization m ends with all spins plus for $p>p_c=2/3$ (solid), $p=p_c$ (dashed), and $p<p_c$ (dotted). Also indicated is the N dependence of the deviation of the first and last curves from a step function.

and after some straightforward steps, the continuum version of the master equation simplifies to

$$(3p-2)\,Nm\,E'(m)+(4-3p)\,E''(m)=0, \tag{2}$$

where $m=2x-1$ is the magnetization and the prime denotes differentiation with respect to m. This equation can be easily integrated and the final result is

$$E(m)=\frac{1}{2}\left(1+\frac{I(m)}{I(1)}\right), \tag{3}$$

where

$$I(m)=\int_0^{\sqrt{m}} e^{-N\alpha y^2/2}\,dy,$$

with $\alpha=(3p-2)/(4-3p)$.

The behavior of $E(m)$ versus m is sketched in Fig. 2 and it merely represents the continuum version of the corresponding result given in Ref. [1]. For $p>p_c$, the exit probability approaches a step function as $N\to\infty$ with a characteristic width that scales as $N^{-1/2}$. This feature reflects the fact that when $|m|>N^{-1/2}$, the hopping process underlying the exit probability is controlled by the global bias. Conversely, for $p<p_c$, the exit probability approaches 1/2 for nearly all initial values of m except for a thin region of width e^{-N} about $m=\pm 1$. This reflects the fact that minority rule tends to drive the system toward zero magnetization. Thus the exit probability is independent of the initial state unless the system begins at an exponentially small distance (in N) from consensus.

B. Magnetization

The average magnetization also obeys a simple rate equation in the continuum limit. With probability $3x^2(1-x)$, where $x=n/N$, a group of three consists of 2 plus spins and 1 minus spin. If this group is picked, then majority rule applies with probability p and the magnetization increases by 2, while with probability q, the magnetization decreases by 4. A complementary reasoning applies to a group with 2 minus spins and 1 plus spin. Thus the rate equation for the magnetization is

$$\frac{dm}{dt}=6x^2(1-x)(p-2q)-6x(1-x)^2(p-2q)$$
$$=6(3p-2)\,m(1-m^2), \tag{4}$$

where again $m=2x-1$. This approximate equation becomes an exact description in the limit $N\to\infty$. The long-time solution is

$$m(t)\simeq\begin{cases}\pm\left\{1-\left[\dfrac{1-m^2(0)}{m^2(0)}\right]e^{-36(p-p_c)t}\right\}, & p>p_c\\ m(0), & p=p_c\\ \dfrac{m(0)}{\sqrt{1-m^2(0)}}e^{-18(p_c-p)t}, & p<p_c,\end{cases} \tag{5}$$

where in the first line, the $\pm$ sign occurs if $m(0)>0$ or $m(0)<0$, respectively.

For $p>p_c$, majority rule prevails and the dynamics is essentially the same as in the original majority rule model [1]. The approach to the asymptotic behavior is exponential in time with a relaxation time $\tau_M=[36(p-p_c)]^{-1}$. This corresponds to an exit time that scales logarithmically in the system size. Conversely, when $p<p_c$ the dynamics is dominated by the rule of the minority so that the asymptotic magnetization vanishes [for $m(0)\neq\pm 1$]. The approach towards this steady state is again exponential, but with a relaxation time $\tau_m=[18(p_c-p)]^{-1}$ that is twice as large as τ_M. In spite of the bias away from consensus, this state is necessarily reached in a finite system, because this is the only absorbing state of the dynamics, but the time required to reach consensus grows exponentially in the system size. Finally, at the critical point $p_c=2/3$, the average magnetization remains invariant, as in the voter model [3].

III. MM MODEL IN ONE DIMENSION

A. Equations of motion

In one dimension, the original formalism of the Ising-Glauber model [4] can be exploited to obtain the equation of motion for the magnetization, as well as that for higher-order spin correlation functions. We consider only the simplest case of group size equal to 3 and denote the spins in a group, which can take the values ± 1, by S, S', and S''. Then the rate at which spin S flips according to majority rule is [1]

$$W(S\to -S)=1+S'S''-S(S'+S''). \tag{6}$$

This rate expresses the fact that S' and S'' must be equal but opposite to S for spin S to flip. Conveniently, this same expression also gives the rate at which the spins S' and S'' flip according to minority rule dynamics. Thus for minority rule the spin-flip rate $w(S',S''\to -S',-S'')\equiv w(S',S'')=W(S\to -S)$.

First consider majority rule dynamics. In this case a given spin S_j belongs to the three groups (S_{j-2},S_{j-1},S_j), (S_{j-1},S_j,S_{j+1}), and (S_j,S_{j+1},S_{j+2}). This then leads to the total flip rate [1]

$$W(S_j\to -S_j)=3+S_{j-2}S_{j-1}+S_{j-1}S_{j+1}+S_{j+1}S_{j+2} -S_j[2S_{j-1}+2S_{j+1}+S_{j-2}+S_{j+2}]. \quad (7)$$

On the other hand, for minority rule, the spin-flip rates are

$$w(S_{j-2},S_{j-1})=1+S_{j-1}S_{j-2}-S_j(S_{j-1}+S_{j-2}),$$
$$w(S_{j-1},S_{j+1})=1+S_{j-1}S_{j+1}-S_j(S_{j-1}+S_{j+1}),$$
$$w(S_{j+1},S_{j+2})=1+S_{j+1}S_{j+2}-S_j(S_{j+1}+S_{j+2}). \quad (8)$$

The kinetics of the system is described by the master equation for the probability distribution for a given spin configuration $\{S\}$. The derivation of this master equation is standard but tedious and the details are given in Appendix A. From the master equation, we can then compute the rate equation for the magnetization [Eq. (A2)]. For the present discussion, we only study a spatially homogeneous system. In this case Eq. (A2) simplifies considerably and the resulting rate equation is

$$\frac{dm_1(t)}{dt}=6\,(3p-2)\,[m_1(t)-m_3(t)], \quad (9)$$

with the magnetization $m_1(t)\equiv\langle S_j(t)\rangle$ written as the first moment of the spin expectation value, and $m_3(t)\equiv\langle S_j(t)S_{j+1}(t)S_{j+2}(t)\rangle$ is the three-spin correlation function.

Notice that this equation has a very similar structure to Eq. (4), the mean-field equation for the magnetization. In fact, Eq. (9) reduces to Eq. (4) if we neglect fluctuations and assume that $m_3=m_1^3$. From Eq. (9), we deduce several basic facts.

(1) For $p=p_c=\frac{2}{3}$ and $\forall\ m_1(0)$, the magnetization is conserved. This conservation, valid in all spatial dimensions, relies on the fact that the group size equals 3. Thus at p_c we expect kinetics similar to that in the classical voter model.

(2) For any p, a system that is initially in consensus $[m_1(0)=\pm 1]$ or a system with zero initial magnetization $[m_1(0)=0]$ does not evolve. That is, $m_1(t)=m_1(0)=\pm 1$ in the former case and $m_1(t)=m_1(0)=0$ in the latter.

(3) The magnetization is generally *not* conserved, except for the initial state $m_1(0)=0$ or ± 1. This nonconservation leads to unusual kinetics of the interfaces between regions of plus and minus spins. While these domain walls diffuse if they are widely separated, MM dynamics leads to additional interactions between walls when their distance is less than or equal to 2.

(4) For $p\neq p_c$, the equation for the magnetization is not closed but involves the three-spin correlation function. In turn, the equation for this correlation function involves higher-order correlations, thus giving rise to an insoluble, infinite equation hierarchy.

To make analytical progress for the behavior of the magnetization in one dimension, we need to truncate this equation hierarchy. In the following section, we implement such a truncation within the Kirkwood approximation scheme.

B. Kirkwood approximation for the final magnetization

We now study the behavior of the magnetization in one dimension. Contrary to the case of spatial dimension $d>1$, the magnetization quickly approaches a saturation value that has a smooth and nontrivial dependence on $m_1(0)$ [1]. We implement a Kirkwood decoupling scheme to the exact master equation to obtain the mean magnetization $m(t)$. We shall see that this uncontrolled approximation gives surprisingly accurate results.

Our approach is based on writing the exact equation of motion for $m_2(t)=\langle S_j(t)S_{j+1}(t)\rangle$ and then, in the spirit of the Kirkwood approximation [20], factorizing the four-point functions that appear in this equation as products of two-point functions. Such an approach has proven quite successful in a variety of applications to reaction kinetics [22–24]. By solving the resulting nonlinear but closed equation, we obtain an approximate expression for m_2. Then in Eq. (9) for the magnetization, we factorize the three-point function m_3 as m_1m_2 (instead of m_1^3 as in the usual mean-field analysis).

We now determine the equation of motion for the correlation function m_2 from the master equation (A1). Following the same steps as those followed to find the equation for the mean magnetization, we find, after a number of straightforward steps (see Appendix B),

$$\frac{dm_2}{dt}=4[2(1+p)-(4+p)m_2+p(c_2+c_3)] +4(2-3p)\,m_4, \quad (10)$$

where we have used the shorthand notations (for a translationally invariant system) $c_r(t)\equiv\langle S_j(t)S_{j+|r|}(t)\rangle$ and $m_4(t)\equiv\langle S_j(t)S_{j+1}(t)S_{j+2}(t)S_{j+3}(t)\rangle$. In general, we reserve the notation m_{2k} to denote the average value of a chain of $2k$ contiguous spins and c_r for the correlation function between two spins that are separated by a distance r. Thus when the separation between the two spins equals 1, we have that $c_1(t)=m_2(t)\equiv\langle S_j(t)S_{j\pm 1}(t)\rangle$.

In spite of the fact that Eq. (10) is exact, the two-spin correlation function $c_r(t)$ is coupled to higher-order correlations and it is therefore difficult to compute these quantities exactly. However at $p_c=\frac{2}{3}$, this equation is closed in that it involves two-spin correlation functions only (see Sec. III C). For $p\neq p_c$ we simply write m_4 as m_2^2 in Eq. (10), following the Kirkwood approximation. Since we are mainly interested in the stationary state at $t=\infty$, where the variation in the two-point function as a function of r is weak, we also make the assumption that $c_2\approx c_3\approx m_2$.

We show in Sec. III D that this approximation is accurate for the voter model limit of $p=p_c$ and our numerical results also show that this approximation continues to give a reasonable description for the properties of the final state when $p\approx p_c$. It is true, however, that this approach does not provide a good description of the time dependence of the magnetization.

With these approximations and for $p\neq\frac{2}{3}$, Eq. (10) becomes

$$\frac{dm_2}{dt}=4(2-3p)\left[(m_2-1)\left(m_2-\frac{2(1+p)}{2-3p}\right)\right]. \quad (11)$$

Equation (11) admits $m_2(\infty)=1$ as the unique and physically acceptable fixed point. [The other fixed points are

$$m_2^*=\frac{2(1+p)}{2-3p}>1$$

for $0<p<\frac{2}{3}$ and

$$m_2^*=\frac{2(1+p)}{2-3p}<-1$$

for $p>\frac{2}{3}$.] The general solution to Eq. (11), for $0<p\leqslant 1$ and $p\neq p_c=\frac{2}{3}$, is

$$m_2(t)=\frac{A+\beta e^{-20pt}}{A-e^{-20pt}}, \quad (12)$$

where

$$\beta=\frac{p_c(1+p)}{p-p_c} \quad \text{and} \quad A=\frac{m_2(0)+\beta}{m_2(0)-1}.$$

At $p_c=\frac{2}{3}$, we obtain $m_2(t)=1-[1-m_2(0)]e^{-40t/3}$. Thus, for all p, $m_2(t)\to 1$ as $t\to\infty$.

We now exploit this result to compute the final magnetization. In the exact equation (9) for m_1, we write m_3 as m_1m_2 to give

$$\frac{dm_1}{dt}=6(3p-2)m_1(1-m_2). \quad (13)$$

Notice a crucial difference between this equation of motion and the mean-field equation (4). In the stationary state, Eq. (13) predicts that either $m_1(\infty)=0$ or $m_2(\infty)=1$. Since $m_2(t)\neq m_1(t)^2$ in the Kirkwood approximation, this means that $m_1(\infty)$ can be a nontrivial function, even if $m_2(\infty)=1$.

Integrating Eq. (13) gives the formal expression for the final magnetization,

$$m_1(\infty)=m_1(0)\exp\left\{18(p-p_c)\int_0^\infty dt'[1-m_2(t')]\right\}. \quad (14)$$

Substituting the expression for $m_2(t)$ in Eq. (12), we thereby obtain

$$m_1(\infty)=m_1(0)\left(\frac{\beta+1}{\beta+m_2(0)}\right)^{3/2}. \quad (15)$$

For an initially uncorrelated and random system, $m_2(0)=m_1(0)^2$, and

$$m_1(\infty)=m_1(0)\left(\frac{\beta+1}{\beta+m_1(0)^2}\right)^{3/2}. \quad (16)$$

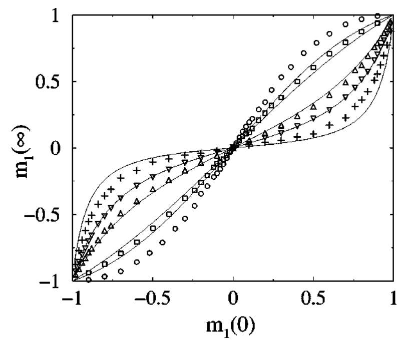

FIG. 3. The final magnetization as a function of the initial magnetization. Shown are the results of numerical simulations for the cases $p=0.1(+)$, $0.25(\nabla)$, $0.4(\Delta)$, $0.8(\square)$, and 1 ($\bigcirc$). The smooth curves are the corresponding results from the Kirkwood approximation [Eq. (16)].

Thus the Kirkwood approximation predicts a final magnetization that is a nontrivial function of the initial magnetization (Fig. 3). As $p\to p_c=\frac{2}{3}$ this approximation correctly predicts that the average magnetization is conserved, that is, $m_1(t)=m_1(0)$. When $p\to 0$, this approximation also predicts [for $m_1(0)\neq\pm 1$] that the final magnetization vanishes [i.e., $m_1(\infty)=0$]. Figure 3 shows that the Kirkwood approximation is quantitatively accurate for intermediate values of p but is only qualitative for p close to either 0 or 1.

C. Two-spin correlation function at p_c

At $p_c=\frac{2}{3}$, Eq. (9) shows that the magnetization of the MM is conserved. This same conservation law occurs in the voter model which has a consequence that the correlation function $\langle S_j(t)S_{j+r}(t)\rangle-1$ vanishes as $t^{-1/2}$ in one dimension. We now show that this same type of coarsening also occurs in the MM by computing the two-spin correlation functions at p_c. The equations of motion for these correlation functions are cumbersome and they are written in Appendix B.

For our purposes, we concentrate on translationally invariant and symmetric systems. Then in the equations of motion for the two-point function [Eqs. (B1)–(B3)], the coordinates j_1,j_2,j_3,j_4 in the four-point functions always appear as three consecutive positions, then a gap of size r, followed by the coordinate of the last spin. This gap can either occur on the left or the right side of the spin group. To simplify the notation, we therefore write these four-point "gap" functions of the form $\langle S_{j-2}(t)S_{j-1}(t)S_j(t)S_{j+r}(t)\rangle$ and $\langle S_j(t)S_{j+r}(t)S_{j+r+1}(t)S_{j+r+2}(t)\rangle$ as $\mathcal{G}_r(t)$. With these simplifications Eq. (B1) gives, for the case of majority rule (i.e., $p=1$) and for $|r|>2$,

$$\frac{dc_r}{dt}=4(c_{r+2}+c_{r-2}+2c_{r+1}+2c_{r-1}-3c_r)$$
$$-4(\mathcal{G}_r+\mathcal{G}_{r-1}+\mathcal{G}_{r-2}). \quad (17)$$

For $r=1$, Eq. (B2) gives

$$\frac{dc_1}{dt}=4(c_2+c_3+4-5c_1-\mathcal{G}_1), \quad (18)$$

while Eq. (B3) gives

$$\frac{dc_2}{dt}=4(2+c_1+2c_3+c_4-4c_2)-4(\mathcal{G}_1+\mathcal{G}_2). \qquad (19)$$

Together with $c_0(t)=1$, Eqs. (17)–(19) are the equations of motion for $c_r(t)$ for the translationally invariant majority model.

For the minority model ($p=0$) we proceed in a similar manner to write the analog of Eq. (B1). After straightforward but lengthy computations the equation of motion is (for $|r|>2$)

$$\frac{dc_r}{dt}=8(\mathcal{G}_r+\mathcal{G}_{r-1}+\mathcal{G}_{r-2})-24c_r, \qquad (20)$$

while for $r=1$

$$\frac{dc_1}{dt}=8(1-2c_1+\mathcal{G}_1), \qquad (21)$$

and for $r=2$ we have

$$\frac{dc_2}{dt}=8(1-3c_2+\mathcal{G}_1+\mathcal{G}_2). \qquad (22)$$

These equations again have to be supplemented by the boundary condition $c_0(t)=1$.

The equation of motion for the two-spin correlation function in the MM model can now be obtained by taking p times Eq. (17) and $1-p$ times Eq. (20). For general p, this leads to an open equation hierarchy. However, at $p=p_c=\frac{2}{3}$, the four-spin correlation functions arising from both the majority and minority models *cancel* for all values of r. Thus at $p=p_c$, we obtain much simpler equations of motion for the two-spin correlation function. For $|r|>2$, we obtain

$$\frac{dc_r}{dt}=\frac{8}{3}[c_{r+2}+c_{r-2}+2(c_{r+1}+c_{r-1})-6c_r]. \qquad (23)$$

For $r=1$ we obtained previously [Eq. (10)]

$$\frac{dc_1}{dt}=\frac{8}{3}(5+c_2+c_3-7c_1), \qquad (24)$$

while for $r=2$ we have

$$\frac{dc_2}{dt}=\frac{8}{3}(3+c_4+2c_3+c_1-7c_2). \qquad (25)$$

Equations (23)–(25), together with the boundary condition $c_0(t)=1$, constitute a closed and soluble set of coupled linear differential-difference equations for the two-spin correlation functions.

D. Solution for the two-spin correlation function

To solve Eqs. (23)–(25), first notice that these coupled equations can be recast as the single equation for the auxiliary quantity $g_r(t)\equiv c_r(t)-1$,

$$\begin{aligned}\frac{dg_r}{dt}=&g_{r+2}+g_{r-2}+2(g_{r+1}+g_{r-1})-6g_r\\&-(g_2+2g_1)(\delta_{r,1}+\delta_{r,-1})-(g_1+g_2)(\delta_{r,2}+\delta_{r,-2})\\&-2(g_2+2g_1)\delta_{r,0}, \end{aligned} \qquad (26)$$

where for simplicity we have also rescaled the time according to $t\to\frac{8}{3}t$.

Before proceeding, it is instructive to recall that in the one-dimensional voter model, the equation for the two-spin correlation function $c_r^{vm}(t)$ for a translationally invariant system has the form of the discrete diffusion equation [4]

$$\frac{d}{dt}c_r^{vm}=c_{r+1}^{vm}+c_{r-1}^{vm}-2c_r^{vm} \qquad (27)$$

for $|r|\geqslant 1$, supplemented by the boundary condition $c_0^{vm}(t)=1$. The solution to this equation is

$$c_r^{vm}(t)=1+e^{-2t}\sum_{l=1}^{\infty}[c_l^{vm}(0)-1][I_{r-l}(2t)-I_{r+l}(2t)], \qquad (28)$$

where $I_r(t)$ is the modified Bessel function of first kind [25].

For the MM model, Eq. (26) is also a discrete diffusion equation but with second-neighbor hopping. Thus we expect that this equation can be solved by similar techniques as those used in the voter model. Therefore we introduce the following integral representation that generalizes the modified Bessel function of the first kind:

$$\mathcal{I}_r(t)\equiv\frac{1}{\pi}\int_0^{\pi}dq\cos(qr)e^{2t[\cos 2q+2\cos q]}. \qquad (29)$$

It is easy to check that $\mathcal{I}_r(t)$ satisfies the basic recursion property $\dot{\mathcal{I}}_r(t)=\mathcal{I}_{r-2}(t)+\mathcal{I}_{r+2}(t)+2[\mathcal{I}_{r-1}(t)+\mathcal{I}_{r+1}(t)]$. Also in analogy with the modified Bessel function of first kind, $\sum_{r=-\infty}^{+\infty}\mathcal{I}_r(t)=e^{6t}$ and $\mathcal{I}_r(0)=\delta_{r,0}$. With these properties, the formal solution of Eq. (26) is

$$\begin{aligned}g_r(t)=&e^{-6t}\sum_{r'=-\infty}^{+\infty}g_{r'}(0)\mathcal{I}_{r-r'}(t)\\&-\int_0^t dt'g_1(t-t')e^{-6t'}\left[4\mathcal{I}_r(t')+2\sum_{r_1}\mathcal{I}_{r_1}(t')+\sum_{r_2}\mathcal{I}_{r_2}(t')\right]\\&-\int_0^t dt'g_2(t-t')\,e^{-6t'}\left[2\mathcal{I}_r(t')+\sum_{r_{1,2}}\mathcal{I}_{r_{1,2}}(t')\right], \end{aligned} \qquad (30)$$

where in the second line the sums are over the nearest and next-nearest neighbors of r, respectively, while in the third line the sum is over both nearest and next-nearest neighbors.

Since the right-hand side of Eq. (30) still depends on g_1 and g_2, we have to consider the cases $r=1$ and 2 separately to obtain the general solution. This is done in Appendix C by using Laplace transforms. In the long-time and large-distance limit, the full solution to Eq. (30) quoted in Eq. (C9) reduces to the much simpler expression

$$c_r(t) \simeq m_1(0)^2 + [1-m_1(0)^2] \ \ \mathrm{erfc}\left(\frac{r}{8\sqrt{t}}\right) \tag{31}$$

that clearly shows the scaling behavior in r and t. For comparison, the two-spin correlation function of the voter model, in the same limit and with the same initial condition of $c_r^{vm}(0)=m_0^2$, is

$$c_r^{vm}(t) \simeq m_0^2 - \frac{(1-m_0^2)}{2\sqrt{\pi t}} \sum_{1\leq l \leq 2r} e^{-(r-l)^2/4t}. \tag{32}$$

Comparing these two results, we see that the MM model shares many of the asymptotic features of the voter model. The correlation between spins that are separated by a fixed distance r both approach the value 1, with the deviation from the asymptotic value decaying as $t^{-1/2}$. As in the voter model, the density of domain walls between regions of plus and minus spins, that is, $[1-c_1(t)]/2$ decays as $t^{-1/2}$ [see Eq. (C8)]. Thus in the one-dimensional MM there is coarsening with typical domains growing as $t^{1/2}$, as in the voter model [26].

Our exact results also shed light on the basic nature of the Kirkwood approximation. This approximation gave $c_1(t) = 1-[1-m_1(0)^2]e^{-40t/3}$, whereas the exact result of Eq. (C8) predicts that $c_1(t)$ approaches 1 with a correction term proportional to $t^{-1/2}$. Although both expressions give the same asymptotic state of consensus, the incorrect time dependence in the Kirkwood approximation appears to stem from our assumption that $c_1(t) \approx c_2(t) \approx c_3(t)$. Although this is valid in the stationary state, it is certainly incorrect in the transient regime where this assumption is at odds with the diffusive nature of the problem. As confirmed by numerical results, we thus expect that the Kirkwood approximation should give good results for the stationary magnetization, but not for the approach to this state.

IV. SUMMARY AND DISCUSSION

We introduced a simple model of opinion dynamics—termed the MM model—in which a fixed-size group of agents is specified and all members of the group adopt the local majority state with probability p or the local minority state with probability $1-p$. We considered the simplest case where the group size $G=3$. In the mean-field limit, the probability that the system ends with all spins plus as a function of the initial magnetization of the system (the exit probability) can be readily obtained. For $p > p_c = \frac{2}{3}$, this exit probability changes abruptly from -1 for initial magnetization $m(0) < 0$ to $+1$ for $m(0) > 0$. Conversely, for $p < p_c$, this exit probability is 1/2 for almost all $m(0)$. These behaviors reflect the inherent biases of majority and minority rules.

In one dimension, the magnetization quickly approaches a fixed value that depends only on the initial magnetization. This then immediately determines the exit probability. Within a Kirkwood decoupling scheme for the infinite hierarchy of equations for correlation functions, we obtained a reasonable approximation for the dependence of the final magnetization (equivalently the exit probability) on the initial magnetization. It is worth noting that other decoupling schemes can also be applied. One such example is the so-called "simple method" [27], where the three- and four-point correlation functions are decoupled according to $m_3 = m_2^2/m_1$ and $m_4 = m_2^2$. While this approach sometimes gives superior results to the Kirkwood scheme [23], this approach turns out to be ill suited to determining the initial density dependence of the final magnetization in one dimension.

At the critical point of $p_c = \frac{2}{3}$, we obtained the exact two-spin correlation function and showed that it exhibits the same $t^{1/2}$ coarsening as in the classical voter model. Although the two-spin correlation function has the same behavior as in the voter model, it is possible that two-time correlation functions, such as $\langle S(t)S(t')\rangle$, or quantities related to persistence phenomena, will give behavior different than the voter model.

We would like to suggest several directions for further research. First, it would be worthwhile to understand the MM model in finite spatial dimensions strictly greater than 1. In the special case where the majority exclusively rules ($p=1$), numerical evidence suggested that the upper critical dimension of the system is greater than 4 [1]. On the other hand, the upper critical dimension for the voter model equals 2 and this appears to coincide with the behavior of the MM model for $p=p_c$. It should be instructive to understand the nature of the crossover between these two behaviors.

Another question involves the dependence of the kinetics on the group size. For group size $G>3$, a sharp transition between majority-dominated and minority-dominated kinetics can be engineered by the following somewhat baroque construction. For a group that contains k plus spins and $G-k$ minus spins, apply majority rule with probability k/G and minority rule with probability $1-k/G$. It is easy to verify that this rule gives zero net magnetization change in each elemental group update. Thus this construction should lead to kinetics similar to that of the voter model. However, in the more natural situation where the probabilities of applying the majority or minority rules are independent of group composition, we do not yet understand the nature of the change between majority-dominated and minority-dominated dynamics.

The kinetics in one dimension presents an intriguing challenge. Within the Glauber formalism, the MM model appears to be insoluble because correlation functions of different orders are coupled in the equations of motion. However, the evolution of interfaces between domain walls obeys relatively simple kinetics that closely resembles the diffusion-limited reaction $A+A \rightarrow 0$. For the MM model, it is easy to see that, in addition to diffusion of domain walls, there are specific constraints in their motion when domain walls are either nearest-neighbor or next-nearest neighbor. In spite of these complications, we would hope that this model is exactly soluble in one dimension.

Finally, it should be worthwhile to extend the model to allow for agents that have an intrinsic identity. In the MM model, the state of an agent is determined only by the local environment. However, it is much more realistic for individuals to inherently prefer one of the two states so that the transition rates depend both on this factor as well as on the state of its neighbors. This seems a natural step to bring the MM model a bit closer to political reality.

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APPENDIX A: MASTER EQUATION

We write the master equation for the probability distribution of a given spin configuration and then use this to obtain the equation of motion for the magnetization. From the definition of the MM, the master equation is

$$\begin{aligned}\frac{d}{dt}P(\{S\},t)=p\sum_k\,[W(-S_k\to S_k)P(\{S\}_k,t)-W(S_k\to -S_k)P(\{S\},t)]\\+(1-p)\sum_k\,[w(-S_{k-2};-S_{k-1})P(\{S\}_{k-2,k-1},t)-w(S_{k-2};S_{k-1})P(\{S\},t)]\\+(1-p)\sum_k\,[w(-S_{k-1};-S_{k+1})P(\{S\}_{k-1,k+1},t)-w(S_{k-1};S_{k+1})P(\{S\},t)]\\+(1-p)\sum_k\,[w(-S_{k+1};-S_{k+2})P(\{S\}_{k+1,k+2},t)-w(S_{k+1};S_{k+2})P(\{S\},t)].\end{aligned}\tag{A1}$$

Here $P(\{S\},t)$ denotes the probability for the spin configuration $\{S\}$ at time t and $P(\{S\}_k,t)$ is the probability for the configuration $\{S\}_k$ where spin S_k is reversed compared to $\{S\}$. Similarly $P(\{S\}_{k_1,k_2},t)$ is the probability of the configuration where spins S_{k_1} and S_{k_2} are reversed compared to $\{S\}$.

From this master equation, and with the help of Eqs. (7) and (8), it follows that the mean magnetization obeys the equation of motion

$$\begin{aligned}\frac{d}{dt}\langle S_j\rangle&=\sum_{\{S\}}\,S_j\frac{d}{dt}P(\{S\},t)\\&=2p[\langle S_{j-2}\rangle+\langle S_{j+2}\rangle+2\langle S_{j-1}\rangle+2\langle S_{j+1}\rangle-3\langle S_j\rangle]\\&\quad-2p[\langle S_jS_{j+1}S_{j+2}\rangle+\langle S_{j-1}S_jS_{j+1}\rangle+\langle S_{j-2}S_{j-1}S_j\rangle]\\&\quad-2(1-p)[6\langle S_j\rangle-2\langle S_{j-2}S_{j-1}S_j\rangle-2\langle S_{j-1}S_jS_{j+1}\rangle-2\langle S_jS_{j+1}S_{j+2}\rangle].\end{aligned}\tag{A2}$$

To arrive at this equation, we have taken the thermodynamic limit, made some obvious cancellations, and used the following relations:

$$\sum_{\{S\}}\,S_jP(\{S\}_j)=-\langle S_j\rangle;\quad \sum_{\{S\}}\,S_jP(\{S\}_{k\neq j})=\langle S_j\rangle,$$

$$\sum_{\{S\}}\,S_jP(\{S\}_{j,k'\neq j})=-\langle S_j\rangle;\quad \sum_{\{S\}}\,S_jP(\{S\}_{k\neq j,k'\neq j})=\langle S_j\rangle,$$

$$\sum_{\{S\}}\,S_jS_{j'}P(\{S\}_{j,j'})=\langle S_jS_{j'}\rangle$$

APPENDIX B: EQUATIONS OF MOTION FOR TWO-SPIN CORRELATION FUNCTIONS

We write the general equations of motion for the two-spin correlation functions. For simplicity consider the case of majority rule (i.e., $p=1$). In this case, we have

$$\begin{aligned}\frac{d}{dt}\langle S_j(t)S_{j+r}(t)\rangle &= -2\langle S_jS_{j+r}W(S_j\to -S_j)\rangle - 2\langle S_j S_{j+r}\, W(S_{j+r}\to -S_{j+r})\rangle \\ &= 2[\langle S_{j-2}(t)S_{j+r}(t)\rangle + \langle S_{j+2}(t)S_{j+r}(t)\rangle + 2\langle S_{j-1}(t)S_{j+r}(t)\rangle + 2\langle S_{j+1}(t)S_{j+r}(t)\rangle - 6\langle S_j(t)S_{j+r}(t)\rangle] \\ &\quad + 2[\langle S_j(t)S_{j+r-2}(t)\rangle + \langle S_j(t)S_{j+r+2}(t)\rangle + 2\langle S_j(t)S_{j+r-1}(t)\rangle + 2\langle S_j(t)S_{j+r+1}(t)\rangle] \\ &\quad - 2[\langle S_{j-2}(t)S_{j-1}(t)S_j(t)S_{j+r}(t)\rangle + \langle S_{j-1}(t)S_j(t)S_{j+1}(t)S_{j+r}(t)\rangle + \langle S_j(t)S_{j+1}(t)S_{j+2}(t)S_{j+r}(t)\rangle] \\ &\quad - 2[\langle S_j(t)S_{j+r-2}(t)S_{j+r-1}(t)S_{j+r}(t)\rangle + \langle S_j(t)S_{j+r-1}(t)S_{j+r}(t)S_{j+r+1}(t)\rangle \\ &\quad + \langle S_j(t)S_{j+r}(t)S_{j+r+1}(t)S_{j+r+2}(t)\rangle]. \end{aligned} \tag{B1}$$

This equation applies for $r\neq 0,\pm 1,\pm 2$. For $r=0$ we have simply $\langle S_j(t)^2\rangle = 1$. The cases $r=\pm 1$ and $r=\pm 2$ have to be dealt with separately. For $r=1$, we have

$$\begin{aligned}\frac{d}{dt}\langle S_j(t)S_{j+1}(t)\rangle &= 2[8 + \langle S_{j-2}(t)S_{j+1}(t)\rangle - 10\langle S_j(t)S_{j+1}(t)\rangle + \langle S_j(t)S_{j+2}(t)\rangle \\ &\quad + \langle S_{j-1}(t)S_{j+1}(t)\rangle + \langle S_j(t)S_{j+3}(t)\rangle] \\ &\quad - 2[\langle S_{j-2}(t)S_{j-1}(t)S_j(t)S_{j+1}(t)\rangle + \langle S_j(t)S_{j+1}(t)S_{j+2}(t)S_{j+3}(t)\rangle] \end{aligned} \tag{B2}$$

and the equation $r=-1$ has a very similar form. For $r=2$ we obtain

$$\begin{aligned}\frac{d}{dt}\langle S_j(t)S_{j+2}(t)\rangle &= 2[4 - 8\langle S_j(t)S_{j+2}(t)\rangle + 2\langle S_j(t)S_{j+3}(t)\rangle + 2\langle S_{j-1}(t)S_{j+2}(t)\rangle + \langle S_j(t)S_{j+4}(t)\rangle] \\ &\quad + 2[\langle S_{j-2}(t)S_{j+2}(t)\rangle + \langle S_j(t)S_{j+1}(t)\rangle + \langle S_{j+1}(t)S_{j+2}(t)\rangle] \\ &\quad - 2[\langle S_{j-2}(t)S_{j-1}(t)S_j(t)S_{j+2}(t)\rangle + \langle S_{j-1}(t)S_j(t)S_{j+1}(t)S_{j+2}(t)\rangle] \\ &\quad - 2[\langle S_j(t)S_{j+1}(t)S_{j+2}(t)S_{j+3}(t)\rangle + \langle S_j(t)S_{j+2}(t)S_{j+3}(t)S_{j+4}(t)\rangle], \end{aligned} \tag{B3}$$

and similarly for $r=-2$. For a translationally invariant system, Eqs. (B1)–(B3) reduce, respectively, to Eqs. (17)–(19).

The equations of motion for minority rule (where $p=0$) are obtained in a similar manner by starting with the analog of Eq. (B1) when the minority rule hopping rates are used. For the translationally invariant minority model, the equations of motion for the correlation functions are then given by Eqs. (20)–(22).

APPENDIX C: SOLUTION FOR THE CORRELATION FUNCTION

In this appendix, we solve Eq. (30). For this purpose it is convenient to introduce the Laplace transform. For an uncorrelated but random initial state where $c_r(0)=m_1(0)^2$, and using the properties of the functions $\mathcal{I}$ introduced in Eq. (29), the Laplace transform of $g_r(t)$ is

$$\begin{aligned}\hat{g}_r(s) &\equiv \int_0^\infty dt\, e^{-st} g_r(t) \\ &= -\frac{1-[m_1(0)]^2}{s} - [4\hat{\mathcal{I}}_r(s) + 2\hat{\mathcal{I}}_{r+1}(s) + 2\hat{\mathcal{I}}_{r-1}(s) + \hat{\mathcal{I}}_{r+2}(s) + \hat{\mathcal{I}}_{r-2}(s)]\hat{g}_1(s) \\ &\quad - [2\hat{\mathcal{I}}_r(s) + \hat{\mathcal{I}}_{r+1}(s) + \hat{\mathcal{I}}_{r-1}(s) + \hat{\mathcal{I}}_{r+2}(s) + \hat{\mathcal{I}}_{r-2}(s)]\hat{g}_2(s), \end{aligned} \tag{C1}$$

where

$$\begin{aligned}\hat{\mathcal{I}}_r(s) &\equiv \int_0^\infty dt\, e^{-st}[e^{-6t}\mathcal{I}_r(t)] \\ &= \int_0^\pi \frac{dq}{\pi} \frac{\cos qr}{s+6-2\{\cos 2q + 2\cos q\}} \\ &= i\oint_\Gamma \frac{dz}{2\pi} \frac{z^{r+1}}{z^4+2z^3-(s+6)z^2+2z+1}, \end{aligned} \tag{C2}$$

and Γ denotes the unit circle in the complex plane centered at the origin. In principle, integral (C2) can be computed by the residue theorem. However, we shall see that this calculation is unnecessary for determining the long-time behavior of the correlation functions.

By substituting $r=1$ and $r=2$ into Eq. (C1) we obtain a linear system of two equations that is readily solved and gives, for the Laplace transforms of $g_1(t)$ and $g_2(t)$,

$$\hat{g}_1(s)=\{1-[m_1(0)]^2\}\frac{\mathcal{J}_2(s)-\mathcal{K}_2(s)-1}{[1+\mathcal{J}_1(s)+\mathcal{K}_2(s)-\mathcal{J}_2(s)\mathcal{K}_1(s)+\mathcal{J}_1(s)\mathcal{K}_2(s)]s},$$

$$\hat{g}_2(s)=\{1-[m_1(0)]^2\}\frac{\mathcal{K}_1(s)-\mathcal{J}_1(s)-1}{[1+\mathcal{J}_1(s)+\mathcal{K}_2(s)-\mathcal{J}_2(s)\mathcal{K}_1(s)+\mathcal{J}_1(s)\mathcal{K}_2(s)]s}, \tag{C3}$$

where we have introduced the following quantities:

$$\mathcal{J}_1(s)\equiv 2\hat{\mathcal{I}}_0(s)+5\hat{\mathcal{I}}_1(s)+2\hat{\mathcal{I}}_2(s)+\hat{\mathcal{I}}_3(s),$$

$$\mathcal{K}_1(s)\equiv \hat{\mathcal{I}}_0(s)+3\hat{\mathcal{I}}_1(s)+\hat{\mathcal{I}}_2(s)+\hat{\mathcal{I}}_3(s),$$

$$\mathcal{J}_2(s)\equiv \hat{\mathcal{I}}_0(s)+2\hat{\mathcal{I}}_1(s)+4\hat{\mathcal{I}}_2(s)+2\hat{\mathcal{I}}_3(s)+\hat{\mathcal{I}}_4(s),$$

$$\mathcal{K}_2(s)\equiv \hat{\mathcal{I}}_0(s)+\hat{\mathcal{I}}_1(s)+2\hat{\mathcal{I}}_2(s)+\hat{\mathcal{I}}_3(s)+\hat{\mathcal{I}}_4(s). \tag{C4}$$

Since we are mainly interested in the long-time behavior of the two-spin correlation functions, we focus on the small-s dependence of the quantities in Eq. (C4). For $s\to 0$ integral (C2) diverges for $q\to 0$. Clearly, the main contribution to this integral in the long-time limit (equivalently $s\to 0$) is obtained by expanding the integrand for $q\to 0$ before performing the integration. We obtain

$$\hat{\mathcal{I}}_r(s)\underset{s\to 0}{\to}\int_0^{\pi}\frac{dq}{\pi}\,\frac{\cos(qr)}{s+6q^2}\simeq\int_0^{\infty}\frac{dq}{\pi}\,\frac{\cos(qr)}{s+6q^2}=\frac{e^{-r\sqrt{s/6}}}{2\sqrt{6s}}. \tag{C5}$$

Substituting this expression into Eq. (C3) and expanding the resulting exponential terms gives

$$\hat{g}_1(s)\underset{s\to 0}{\to}-\frac{2}{25}[1-m_1(0)^2]\sqrt{\frac{6}{s}},$$

$$\hat{g}_2(s)\underset{s\to 0}{\to}-\frac{1-m_1(0)^2}{5}\sqrt{\frac{6}{s}}. \tag{C6}$$

The expressions for $\hat{g}_1$ and $\hat{g}_2$, together with Eq. (C1), provide the Laplace transform of $\hat{g}_r(r)$ in the $s\to 0$ regime.

For finite $r>2$, we substitute Eqs. (C5) and (C6) into Eq. (C1), expands the exponential terms as $r\sqrt{s}\to 0$, and obtain

$$\hat{g}_r(s)\underset{s\to 0}{\to}-[1-m_1(0)^2]\frac{5r-4}{5\sqrt{6s}}. \tag{C7}$$

Laplace inverting Eqs. (C6) and (C7) then gives, for $t\to\infty$, with $r^2\ll t$,

$$c_1(t)=1-\frac{3[1-m_1(0)^2]}{25\sqrt{\pi t}},$$

$$c_2(t)=1-\frac{3[1-m_1(0)^2]}{10\sqrt{\pi t}},$$

$$c_r(t)=1-[1-m_1(0)^2]\frac{5r-4}{20\sqrt{\pi t}}\quad(r\geqslant 2), \tag{C8}$$

where we have restored the original time scale, i.e., $t\to\frac{3}{8}t$.

In the limit $r\to\infty$ and $s\to 0$, with $r\sqrt{s}$ kept fixed, we substitute Eqs. (C6) into Eq. (C1), and obtain, after inverse Laplace transforming,

$$c_r(t)=[m_1(0)]^2+\frac{1-[m_1(0)]^2}{50}\left[18\,\mathrm{erfc}\left(\frac{r}{8\sqrt{t}}\right)+7\,\mathrm{erfc}\left(\frac{r+2}{8\sqrt{t}}\right)+7\,\mathrm{erfc}\left(\frac{r-2}{8\sqrt{t}}\right)\right]$$
$$+\frac{9\{1-[m_1(0)]^2\}}{50}\left[\mathrm{erfc}\left(\frac{r+1}{8\sqrt{t}}\right)+\mathrm{erfc}\left(\frac{r-1}{8\sqrt{t}}\right)\right], \tag{C9}$$

where $\mathrm{erfc}(t)\equiv(2/\sqrt{\pi})\int_t^\infty dz\, e^{-z^2}$ is the complementary error function, and we used the fact that the inverse Laplace transform of $e^{-\sqrt{sa}}/s$ is $\mathrm{erfc}(\frac{1}{2}\sqrt{a/t})$ [25]. Equation (C9) simplifies considerably if we make the $r\to\infty$ approximation $r\approx r\pm 1\approx r\pm 2$. In this limit, we obtain the expression quoted in Eq. (31).

[1] P.L. Krapivsky and S. Redner, Phys. Rev. Lett. **90**, 238701 (2003).
[2] S. Galam, Physica A **274**, 132 (1999); Eur. Phys. J. B **25**, 403 (2002).
[3] T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985); T.M. Liggett, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (Springer, Berlin, 1999).
[4] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).
[5] T.J. Cox, Ann. Prob. **17**, 1333 (1989).
[6] P.L. Krapivsky, Phys. Rev. A **45**, 1067 (1992).
[7] S. Galam, Y. Gefen, and Y. Shapir, Math. J. of Soc. **9**, 1 (1982).
[8] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C **11**, 1157 (2000).
[9] F. Slanina, and H. Lavicka, e-print cond-mat/0305102.
[10] A.T. Bernardes, U.M.S. Costa, A.D. Araujo, and D. Stauffer, Int. J. Mod. Phys. C **12**, 159 (2001); D. Stauffer, J. Artifi. Soc. Social Simulation **5**, 1 (2001); See http://www.soc.surrey.ac.uk/JASSS/5/1/4. html (2001); Int. J. Mod. Phys. C **13**, 315 (2002); Adv. Complex Syst. **5**, 97 (2002).
[11] A.A. Moreira, J.S. Andrade, Jr., and D. Stauffer, Int. J. Mod. Phys. C **12**, 39 (2001); D. Stauffer and P.M.C. de Oliveira, Eur. Phys. J. B **30**, 587 (2002).
[12] K. Sznajd-Weron, Phys. Rev. E **66**, 046131 (2002); K. Sznajd-Weron and R. Weron, Int. J. Mod. Phys. C **13**, 115 (2002); K. Sznajd-Weron and R. Weron, e-print cond-mat/0211058
[13] C. Schulze, e-print cond-mat/0207181; e-print cond-mat/0209341; R. Ochrombel, Int. J. Mod. Phys. C **12**, 1091 (2001).
[14] C. Castellano, M. Marsili, and A. Vespignani, Phys. Rev. Lett. **85**, 3536 (2000); D. Vilone, A. Vespignani, and C. Castellano, Eur. Phys. J. B **30**, 399 (2002); K. Klemm, V.M. Eguíluz, R. Toral, and M. San Miguel, Phys. Rev. E **67**, 026120 (2003).
[15] F. Vazquez, P.L. Krapivsky, and S. Redner, J. Phys. A **36**, L61 (2003).
[16] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Physica D **183**, 190 (2003).
[17] W. Weidlich, *Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences* (Harwood Academic, Chur, Switzerland, 2000); *Modeling Complexity in Economic and Social Systems*, edited by F. Schweitzer (World Scientific, Singapore, 2002).
[18] P. Chen, M. Mobilia, and S. Redner (unpublished).
[19] V. Spirin, P.L. Krapivsky, and S. Redner, Phys. Rev. E **63**, 036118 (2001); Phys. Rev. E **65**, 016119 (2002).
[20] See, e.g., R. Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley, New York, 1975); the method was originally introduced in J.G. Kirkwood, J. Chem. Phys. **3**, 300 (1935).
[21] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, 2001).
[22] H. Schnörer, V. Kuzovkov, and A. Blumen, Phys. Rev. Lett. **63**, 805 (1989).
[23] J.C. Lin, C.R. Doering, and D. ben-Avraham, Chem. Phys. **146**, 355 (1990); J.C. Lin, Phys. Rev. A **44**, 6706 (1991).
[24] L. Frachebourg and P.L. Krapivsky, J. Phys. A **31**, L287 (1998).
[25] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1965).
[26] L. Frachebourg and P.L. Krapivsky, Phys. Rev. E **53**, R3009 (1996).
[27] T.R. Waite, Phys. Rev. **107**, 463 (1957); L. Mochik, J.L. Magee, and A.H. Samuel, J. Chem. Phys. **26**, 935 (1957).