Assignment of Probability to the History of Paths in Quantum Phenomena

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Abstract

The question of assignment of probability to path history in quantum phenomena is examined in what are believed to be simple quantum phenomenal structures. The harmonic oscillator and its various ramifications are analyzed in the framework of Complex Measure Theory and it is shown that conditional measure/probability can be assigned to the history of simple paths like transit through Gaussian slits.

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1 Introduction

The object of this contribution is to show that the assignment of probability to paths is a tricky affair and in direct conflict with some of the fundamental postulates of quantum mechanics. Quantum phenomena has been explained all along essentially by Schrödinger-Heisenberg-Dirac approach where probability has practically no role to play. Von Neumann(1955) who provided a mathematical basis for the new mechanics in terms of a Hilbert structure with operators merely crafted the Born interpretation by an additional statement that the trace of an operator corresponds to probability.It is rather strange that Planck radiation formula that has many an implication on statistical aspects is explained away by current formulation of quantum mechanics. However the interpretation of quantum mechanics with specific reference to the notion of an event and its connection to the further evolution of the state vector is tricky and very often it is not recognized that a satisfactory explanation is not within the framework of original postulates. The Feynman path integral formalism1948)is an attempt to overcome some of these difficulties; firmly based on the earlier investigations of Dirac(1933) the formalism is viable enough to translate the classical time evolution in quantum mechanical terms and provides a description in terms of trajectories each of which,by the superposition principle,contributes to the quantum mechanical amplitude. Although it appears very tempting to interpret it as some kind of a complex measure(see for example Gelfand and Yagalom, 1956),it is rather difficult to figure out any kind of positive definite measure in view of the unbounded nature of the variational measure arising therefrom1960)(see however Ito 1961). Nevertheless the path integral approach provides an alternative self-contained reformulation of quantum mechanics (see Feynman and Hibbs 1965)and is capable of direct extension to cover gauge fields. However the probabilistic interpretation is to come through the Born interpretation since neither the classical action nor its quantum mechanical counterpart implies any notion of probability.

It is to be specially noted that the state of a quantum system, the Hilbert space to which it belongs and the measurement system involve the notion of an event at a single epoch and the Born interpretation provides a link, albeit weak, to the probabilistic structure; the non-deterministic elements arising from measurement and events at different epochs cannot be interpreted from the Born interpretation without adding further postulates. Despite this the probabilistic notion corresponding to multiple epochs is freely introduced in the current literature giving the impression that the path integral formalism is all inclusive to admit a comprehensive probabilistic structure. Unfortunately the so called path integral measure is not additive; nor can a bounded variational measure be derived from it. However the experimental results favor a deeper connection to probability(see for example Youssef, 1991,1994);exchanges do continue over quantum phenomena being formulated as a stochastic process(Skorobogatov and Svertilov, 1998). It is in this context that the complex measure theoretic approach proposed some time back(Srinivasan and Sudarshan, 1994;Srinivasan,1997,1998) assumes significance. In such a Complex Measure Theoretic Framework(CMTF),quantum phenomena is modelled as a complex measurable stochastic process endowed with Markov property;some further constraints on the process lead to a Fokker-Planck type equation with complex valued drift and diffusion functions. It turns out that explicit characterization is possible in the case of a free harmonic oscillator and forced harmonic one and this provides a method of dealing with the problem of interaction of radiation with matter; connections are provided to the path integral formalism in a rather gross manner and tally is provided with major results, an interesting feature being the removal of divergences at least in the context of Lamb shift with the ultra-violet cutoff in conventional method now turning out to be valid numerical approximation in CMTF. The object of this contribution to present the analysis of the harmonic oscillator and its various ramifications and examine how far it is possible to associate a positive definite probability to the history of trajectories generated by a harmonic oscillator.

2 Simple harmonic oscillator

At the outset we note that the one dimensional quantum harmonic oscillator is described in CMTF by a diffusion process for the coordinate x with the drift and diffusion functions specified by

(1) $$A(x) = -i\omega x \qquad \text{(Drift)}$$

(2) $$D(x) = -i\hbar/(2m) \qquad \text{(Diffusion)}$$

The complex measure density (CMD)function $\pi(x, t \mid x_{0)}$ of the coordinate is then given by [11)]

(3) $$\pi(x, t \mid x_0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} exp\left\{-\frac{m\omega}{\hbar}\;\frac{(x - x_0 e^{-i\omega t})^2}{(1 - e^{-2i\omega t})}\right\}$$

where x_0 is the position of the coordinate at time $t = o$(initially). It is reasonable to conclude that this corresponds to a measurement leading to the value x_0 at $t = 0$. Under the gimmick $\omega \rightarrow \omega - i\epsilon$, the limit of π as $t$tends to infinity is given by

(4) $$\lim \pi(x, t \mid x_0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega x^2}{\hbar}\right\}$$

so that we conclude that the oscillator relaxes to the ground state ultimately. It is worth noting that the measure density in the limit is real and positive valued quite accidentally. However for a finite value of t, we consider the variational measure density (called modulus measure density in (Srinivasan 1997) appropriately normalized and this is given by

(5) $$\pi(x, t \mid x_0) = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega x^2}{2\hbar}\right\}$$

a result manifestly demonstrating the doubling of the variance as compared to the ground state. As observed earlier(Srinivasan 2001), in CMTF the result obtained from the variational measure can be interpreted as the one resulting from a measurement;thus each time a measurement is made on the coordinate, its variance gets doubled. Thus a sequence of measurements corresponds to the stopping and restarting of the process with the coordinate being governed by the variational density. Hence if a sequence of measurements is made,there results ultimately a real white noise process as is to be expected.

Next we take up the problem of assignment of probability to history. First we note that even for this simple situation there does not exist a stochastic process (with positive definite measure);this is easily verified by taking a multi-time point complex measure density and imposing the modulus on it. Thus it may be worthwhile to consider the ground state and evaluate the the probability of passage of the coordinate through a Gaussian slit. The Gaussian slit is preferred rather than a window of a given width for computational simplicity. Let us assume that the slit is specified by a weight function $f_W(x,\sigma)$ where

$$f_W(x,\sigma) = \left(\frac{1}{(\pi\sigma^2)^{1/2}}\right) \exp\left\{-(x-\overline{x})^2/\sigma^2\right\}. \tag{6}$$

In other words the window is approximated by the function $f_W(x,\sigma)$. At the outset we note that the measure density is independent of t and the probability $p(\overline{x},\sigma,t)$ of passage through the slit at epoch t is given by

$$\begin{aligned} p(\overline{x},\sigma,t) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int f_W(x) \exp\left\{-\frac{m\omega x^2}{\hbar}\right\} dx \\ &= \left\{\frac{1}{\pi}(\sigma^2 + [\hbar/m\omega])\right\}^{1/2} \exp\left\{-\frac{\overline{x}^2}{\sigma^2 + (\hbar/m\omega)}\right\}. \end{aligned} \tag{7}$$

It is interesting to note that the function $p(.,\sigma,t)$ is not a density in as much as the probabilities corresponding to two different but close enough values of $\overline{x}$ are not exclusive; nevertheless the integral of $p(\overline{x},\sigma,t)$ with respect to $\overline{x}$ yields unity by virtue of the special form of the function $f_W(.,\sigma)$. In conventional quantum mechanical approach as contrasted with CMTF, this aspect goes unnoticed in as much as all vectors in the Hilbert space are automatically chosen to have norm equal to unity.[1]We shall presently see the gravity of the situation when we deal with passage through two slits.

[1]However Dowker and Halliwell(1992) did observe the overlapping while introducing projectors for Gaussian slits and had consequently taken into account the overlap while providing constraints for decoherence

Next we assume that the oscillator initially in ground state is passed successively through two slits at epochs t_1 and t_2 respectively centred at x and y with respectively widths σ and ρ. The conditional CMD of the oscillator coordinate at epoch t_2 denoted by $\pi(x_2, t_2 \mid \overline{x}, \sigma, t_1)$[2] is given by

$$\pi(x_2, t_2 \mid \overline{x}, \sigma, t_1) = \left(\frac{A}{\pi}\right)^{1/2} \exp\left\{-A\left[x_2 - \overline{x}\frac{e^{-i\omega(t_2-t_1)}}{[m\omega\sigma^2/\hbar]+1}\right]^2\right\} \tag{8}$$

where

$$A = \frac{m\omega}{\hbar}\frac{1}{1-e^{-2i\omega(t_2-t_1)}/[(m\omega\sigma^2/\hbar)+1]}. \tag{9}$$

From the above expression it is a straightforward exercise to obtain the conditional probability density function of the coordinate $X(t_2)$ at time t_2 conditional on the passage through the slit centered on $\overline{x}$ at timet_1:

$$p(x_2, t_2|\overline{x}, \sigma, t_1) = \left(\frac{m\omega K}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega K}{\hbar}[x_2 - \overline{x}G]^2\right\} \tag{10}$$

where K and G are given by

$$K = \frac{\left(\frac{m\omega\sigma^2}{\hbar}+1\right)\left(\frac{m\omega\sigma^2}{\hbar}+2\sin^2\omega(t_2-t_1)\right)}{\left(\frac{m\omega\sigma^2}{\hbar}+2\sin^2\omega(t_2-t_1)\right)^2 + 4\sin^2\omega(t_2-t_1)\cos^2\omega(t_2-t_1)} \tag{11}$$

$$G = \left(\frac{m\omega\sigma^2}{\hbar}\right)\cos\omega(t_2-t_1)\left(\frac{1}{(\frac{m\omega\sigma^2}{\hbar}+1)(\frac{m\omega\sigma^2}{\hbar}+2\sin^2\omega(t_2-t_1))}\right) \tag{12}$$

Next we note that in order to obtain the conditional probability of passage through another slit $(\overline{y}, \rho)$ at time t_2, we need to multiply $p(x_2, t_2|\overline{x}, \sigma, t_1)$ by the Gaussian window function $f_W(x_2)$ and integrate over x_2. Thus we have

$$p(\overline{y}, \rho, t_2|\overline{x}, \sigma, t_1) = \left(\frac{1}{\pi(\rho^2 + \frac{\hbar}{m\omega K})}\right)^{1/2} \exp\left[-\frac{(\overline{y}-L\overline{x})^2}{\rho^2+\frac{\hbar}{m\omega K}}\right]. \tag{13}$$

Again we note that despite the fact that the probabilities corresponding to two distinct but close enough values of $\overline{y}$ overlap,the function $p(., \rho, t_2|\overline{x}, t_1)$ is normalized in the sense the integral of p over $\overline{y}$ is equal to one. Although the CMD π given by (8) can lead to some kind of a complex probability of passage through the slit,it is not a complex measure in a technical sense and hence no variational measure can be extracted out of it and this is the main reason why we first obtained a positive measure density from π and then

[2]Throughout we use the symbol π to denote the complex measure or measure density and p to denote the positive probability.

proceeded to evaluate the probability of transit through the slit. Thus the question of assignment of consistent probabilities does not arise. In fact the probability of passage through the slit as given by (13) can be used as a crucial test for CMTF itself. Another noteworthy point is that we have a framework to define in a consistent way a sequence of measurements(corresponding,in this case, to a sequence of passages through the slits) since the interference of each measurement is describable as a conditional measure in the first instance.

To sum up the CMD conditional on $X(t) = x_0$ at $t = t_0$, on imposition of modulus measure at $t = t_1$ leads to a zero mean normal variable with variance $\frac{m\omega}{\hbar}$ which is double the value corresponding to the ground state. The modulus measure density is stable in the sense that this density is reproduced every time the process is stopped and restarted with modulus measure density; thus the resulting stochastic process is a collection of i.i.d random variables with a normal density with zero mean and variance equal to $\frac{\hbar}{m\omega}$. It should be noted that these properties hold for short times and that if we wait long enough the system relaxes to the ground state(under the gimmick($\omega \to \omega - i\epsilon$). Even at the risk of repetition,we note that the probability of the passage through the slits is best evaluated by first arriving at the mod measure of the coordinate and then considering the passage through the slit. Next the correlation function $\boldsymbol{E}[X(t_1)X(t_2)]$ with respect to the original complex measure is given by

$$\boldsymbol{E}[X(t_1)X(t_2)] = \left(\frac{\hbar}{2m\omega}\right) e^{-i\omega(t_2-t_1)} \ (t_2 > t_1) \tag{14}$$

If modulus measure is imposed at t_1 and the process re-started,the correlation function is given by

$$\boldsymbol{E}[X(t_1)X(t_2)] = \left(\frac{\hbar}{m\omega}\right) e^{-i\omega(t_2-t_1)} \ (t_2 > t_1). \tag{15}$$

If modulus measure is imposed at the final point t_2, then there results a delta correlation;if modulus measure is imposed at a sequence of epochs, we obtain a white noise. This can be interpreted to mean that too many observations lead to a white noise process implying that there is a quantum phenomenal way of generating the white noise process. The situation changes drastically if we deal with a slightly altered scenario wherein the relevant measure density is genuinely complex valued. We discuss below such a situation where initially the oscillator is conditioned to be in a displaced ground state.

3 Conditioned Harmonic Oscillator

Let us again introduce a free harmonic oscillator when initially (at $t = 0$),the oscillator is in a displaced ground state or equivalently in a coherent state α:

$$\pi(x,t)|_{t=t_0} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega}{\hbar}(x-\alpha)^2\right\} \tag{16}$$

where α is a complex valued parameter. We pass the oscillator through a Gaussian slit$(\overline{x}, \sigma^2)$ at epoch t_1. In this case we define $p(\overline{x}, \sigma, t_1)$ as the normalized modulus value of the complex measure of passing through the slit;now p is given by

$$p(\overline{x},\sigma,t_1) = \left(\frac{1}{\pi([\hbar/m\omega]+\sigma^2)}\right)^{1/2}$$
$$\exp\left\{-\frac{[\overline{x}-\alpha_1 cos\omega(t_1-t_0)-\alpha_2 sin\omega(t_1-t_0)]^2}{[\hbar/m\omega]+\sigma^2}\right\} \tag{17}$$

Next we proceed to obtain the conditional CMD of the oscillator coordinate at t_2, the coordinate process being conditioned by the passage through the slit at t_1; this is conveniently done by first evaluating the CMD of the coordinate at t_2 and the passage through the slit at t_1. Thus we have

$$\pi(x_2, t_2; \text{the process stops and re-starts with mod measure at } t_1\overline{x}, \sigma, t_1; \mid \alpha, t_0)$$
$$= p(\overline{x},\sigma,t_1 \mid \alpha,t_0)\pi(x_2,t_2 \mid \text{mod measure at } t_1; \overline{x},\sigma,t_1;\alpha,t_0) \tag{18}$$

where

$$\pi(x_2,t_2 \mid \text{mod measure at } t_1; \overline{x},\sigma,t_1;\alpha,t_0) = \left(\frac{A}{\pi}\right)^{1/2}$$
$$\exp\left\{-A\left[x_2-\left(\overline{x}+[\alpha_1 cos\omega(t_1-t_0)+\alpha_2 sin\omega(t_1-t_0)]\frac{m\omega}{\hbar}\sigma^2\right)\frac{e^{-i\omega(t_2-t_1)}}{\frac{m\omega}{\hbar}\sigma^2+1}\right]^2\right\} \tag{19}$$

where A is defined by (9). Now the conditional mod measure density function of $X(t_2)$ follows from the above equation; on taking the mod measure and normalizing, we have

$$p(x_2,t_2|\overline{x},\sigma,t_1;\alpha,t_0) = \left(\frac{m\omega K}{\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega K}{\hbar}M^2\right\} \tag{20}$$

where M is given by

$$M = x_2 - G\{\overline{x} + \frac{m\omega\sigma^2}{\hbar}[\alpha_1\cos\omega(t_1-t_0)+\alpha_2\sin\omega(t_1-t_0)]\} \tag{21}$$

where K and G are given respectively by (11) and (12). The conditional probability of passage through a second slit of width ρ centered at $\overline{y}$ is easily obtained; thus we have

$$p(\overline{y},\rho,t_2|\overline{x},\rho,t_1;\alpha,t_0)=\left(\pi\left\{\rho^2+\frac{\hbar}{m\omega K}\right\}\right)^{-1/2}\exp\left\{-\frac{(\overline{y}-H)^2}{\rho^2+\frac{\hbar}{m\omega}K}\right\} \tag{22}$$

where H is given by

$$H=G\left(\overline{x}\frac{m\omega\sigma^2}{\hbar}\{\alpha_1\cos\omega(t_1-t_0)+\alpha_2\sin\omega(t_1-t_0)\}\right). \tag{23}$$

It is worth noting that the joint probability of passage through the two slits can be obtained from (22) and (17) by elementary arguments and the question of consistency does not arise since the passages are defined only sequentially;because of the overlap problem,there is no avenue open in CMTF to define the joint probability directly. Thus unlike conventional quantum mechanical treatment, inconsistencies are automatically avoided here.

At this juncture it is worth looking at the CMTF description of the complex measure of the passage through two slits; since the method gimmicks the conventional quantum mechanical method of analysis,it will be helpful to see the anomaly with greater clarity. Thus if we take the CMD for $X(t_1)$ and integrate it after weighting with the function $f_W(x_1,\rho)$ we have

$$\pi(\overline{x},\sigma,t_1|\alpha,t_0)=\left(\frac{m\omega}{\pi\hbar(\frac{m\omega\sigma^2}{\hbar}+1)}\right)^{1/2} exp\left[-\left\{\frac{[\overline{x}-\alpha e^{-i\omega(t_1-t_0)}]^2}{\frac{\hbar}{m\omega}+\sigma^2}\right\}\right] \tag{24}$$

where it is worth noting that the function $\pi(\overline{x},\sigma,t_1|\alpha,t_0)$ is not a measure density. Anyway we continue in this vein and obtain an expression for the second order complex valued function describing the passage through the slits centered at $(\overline{x},t_1)$ and $(\overline{y},t_2)$:

$$\pi(\overline{y},\rho,t_2;\overline{x},\sigma,t_1|\ \alpha,t_0)=\pi(\overline{x},\sigma^2,t_1|\alpha,t_0)\left\{\frac{1}{\pi}\ \frac{1}{\rho^2+\frac{\hbar}{m\omega}(1-\frac{e^{-2i\omega(t_2-t_1)}}{\frac{m\omega\sigma^2}{\hbar}+1})}\right\}^{1/2}$$

$$exp\left\{-\frac{1}{\rho^2+\frac{\hbar}{m\omega}(1-\frac{e^{-2i\omega(t_2-t_1)}}{\frac{m\omega\sigma^2}{\hbar}+1})}\right\}$$

$$\left[\overline{y}-\alpha e^{-i\omega(t_2-t_0)}-\frac{e^{-i\omega(t_2-t_1)}}{\frac{m\omega}{\hbar}\sigma^2+1}\{\overline{x}-\alpha e^{-i\omega(t_1-t_0)}\}\right]^2 \tag{25}$$

If we now integrate over $\overline{x}$, we find

$$\int \pi(\overline{y},\rho,t_2;\overline{x},\sigma,t_1|\alpha,t_0)d\overline{x} = \left\{\pi(\rho^2+\frac{\hbar}{m\omega})\right\}^{-1/2} exp - \left\{\frac{[\overline{y}-\alpha e^{-i\omega(t_2-t_0)}]^2}{\rho^2+\frac{\hbar}{m\omega}}\right\}$$

$$= \pi(\overline{y},\rho,t_2|\alpha,t_0) \tag{26}$$

a relation showing the normal consistency check for complex measures. However we cannot conclude that the π functions introduced above are genuine measure densities in view of the overlap discussed earlier. Thus we cannot resort to mod measure and normalize to make the total measure unity. At this juncture it is worth noting that in the conventional treatment the density matrix corresponding to the passage through several slits is obtained using the path integral formalism and the desired probability is identified to be the diagonal element of the density matrix. However there is a problem in the conventional quantum mechanical formalism; the Gaussian slit is at best an approximation used to simplify the evaluation of the amplitude/density matrix corresponding to a slit of deterministic width. The passage through the slit amounts to a measurement after all and the Heisenberg principle comes into play at each passage through the slit , a scenario completely ignored. However arguments can be put forward against the applicability of Heisenberg's uncertainty principle by invoking the closed nature of the system; still the normalization factor is to be put in to obtain the probability an aspect,completely ignored in the literature.On the other hand, this issue is automatically well focussed in CMTF as we have seen earlier. In the present case,it is very transparent.In fact this is a fit case for experimental verification. Unfortunately, the Gaussian slit experiment (see for example Feynman and Hibbs (1965)p.47)is still a gadenken.

Last we present one more result that goes to show that theoretically it is possible to arrive at the probability of a two-time 'observation' of the coordinate:

$$\pi(x_2,t_2;x_1,t_1|\alpha,t_0)|_{\text{mod. measure}} = \left(\frac{m\omega}{2\pi\hbar}\right)$$

$$\exp - \left\{\frac{m\omega}{2\hbar}[x_2^2 + (x_1 - 2\alpha_1\cos\omega(t_1-t_0) - 2\alpha_2\sin\omega(t_1-t_0))^2]\right\} \tag{27}$$

Even at the risk of repetition, we observe that no attempt should be made to derive the marginal probability density functions of the coordinate corresponding to time epochs t_1 and t_2 from the above probability density function since in CMTF such density functions essentially arise by measure transformation at these epochs after stopping the process at t_1 and t_2 respectively. From a purely statistical inferential point of view, it is worth noting that a frequency ratio connection can be provided to the joint probability density function given by (26); this will correspond to repeated non-interfering(i.e. non-destructive) type of 'observations' at the epochs t_1 and t_2. In fact there is a generalization of an observation friendly Chebycheff type of inequality derived elsewhere (Srinivasan,2005a)that will provide the desired connection in a limiting sense;the result can be stated as follows:

Extended Chebycheff inequality: If X and Y are two random variables (i.e. measurable functions with respect to the complex measure space)and m_X and m_Y are respectively the expected values with respect to the mod measure,then the complex measure denoted by Pr, for arbitrary $k_1, k_2 > 0$,satisfies

$$|Pr\{|X(t_1) - m_X| > k_1\sigma_X, |X(t_2) - m_Y| > k_2\sigma_Y\} \leq \lambda\kappa/[k_1^2 k_2^2 \sigma_X^2 \sigma_Y^2] \tag{28}$$

where λ is the normalizing constant arising in the definition of mod measure in (26) and κ is the joint moment given by

$$\kappa = \boldsymbol{E}\{[X - m_X]^2[Y - m_Y]^2\} \tag{29}$$

where the expectation is with respect to the mod measure of the joint density function. The proof of the above inequality is fairly straightforward. In the special case when the mod measure density is defined by (26), κ is simply equal to$(1+2\rho^2)\sigma_X^2\sigma_Y^2$. It is interesting to note that the above inequality says nothing about the complex measure or the possible statistics arising therefrom; on the other hand it implies some kind of a convergence of the statistics of the mod measure to those obtained by frequency ratio method.

4 Harmonic oscillator in a bath

Now we wish to discuss the various aspects of a test (harmonic) oscillator in interaction with a collection of oscillators consisting a bath. Normally the study of interacting oscillator is a complex one; however in our case we make the (drastic) bath approximation so that the test oscillator feels the effects of the bath substantially with the bath itself being unaffected by the presence/interaction of the test oscillator. Such an oscillator had been the subject of investigation from various angles by numerous authors including Feynman and Vernon (1963), Ford, Kac and Mazur (1965) and Caldeira and Legget (1983). The model had been elegantly handled by the path integral method by Feynman and Vernon (1963) treating the distinguished (test) oscillator as a forced harmonic oscillator, the forcing term arising from its interaction with the bath. Dowker and Halliwell (1992) had discussed this problem with special reference to the general aspects of quantum mechanical history and decoherence. We here show how certain probabilities relating to the history of paths can be handled in CMTF.

We note that the motion of the distinguished (test) oscillator in interaction with the bath is best visualized as a forced harmonic oscillator with the bath characteristics included in the forcing term. In CMTF, we can introduce the bath characteristics ($\mathbb{B}$) as some kind of a conditioning; denoting the conditional CMD by $\pi(x, t|x_0, t_0; \mathbb{B})$ we note that π satisfies the Fokker-Planck equation with $D(x)$ the usual complex diffusion coefficient and $A(x)$ which is now a function of t also is specified by

$$A(x) = -\mathrm{i}\omega x + \beta(t) \tag{30}$$

$$\beta(t) = \int_0^t \mathrm{e}^{-\mathrm{i}\omega(t-s)} f(s)\mathrm{ds} \tag{31}$$

$$f(t) = -\sum_k C_k R_k(t) \tag{32}$$

The additional term $\beta(t)$ in the drift function $A(x)$ brings out in a transparent manner the linear coupling (of the distinguished oscillator) with the coordinate $R_k(t)$ of the typical oscillator of the bath. Thus the conditional CMD $\pi(x, t|x_0, t_0; \mathbb{B})$ is given by

$$\pi(x, t|x_0, t_0; \mathbb{B}) = \left(\frac{m\omega \mathrm{e}^{2\mathrm{i}\omega t}}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\right)^{1/2} \exp\left\{-\frac{m\omega \mathrm{e}^{2\mathrm{i}\omega t}}{\mathrm{e}^{2\mathrm{i}\omega t} - 1}\left[x - x_0\mathrm{e}^{-\mathrm{i}\omega t} - F(t, t_0)\right]^2\right\} \tag{33}$$

where

$$F(t, t_0) = \frac{1}{\omega}\int_{t_0}^t f(u)\sin\omega(t-u)\,\mathrm{d}u \tag{34}$$

In the ultimate analysis we consider a continuous assembly of oscillators constituting the bath; however to obtain results in a rather simple way, we note that if we deal with one oscillator, then $f(u)$ can be replaced by $-CR(u)$ whose CMD has a Gaussian structure. The process of summing over the different oscillators constituting the bath is analyzed elsewhere(Srinivasan 2005b)); the final result is given by

$$\begin{aligned} \pi_{\text{final}}(x, t|x_0, t_0) &\equiv \mathrm{E}_{\mathbb{B}}[\pi(x, t|x_0, t_0; \mathbb{B})] \\ &= \left\{2\pi\left[\mathcal{A} + \frac{\hbar}{2m\omega}(1 - \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})\right]\right\}^{-1/2} \\ &\exp - \left(x - x_0\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}\right)^2 \Big/ \left[2\mathcal{A} + \frac{\hbar}{m\omega}\left(1 - \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)}\right)\right] \end{aligned} \tag{35}$$

where $\mathrm{E}_{\mathbb{B}}$ denotes the expectation over the bath variables and $\mathcal{A}$ is given by

$$\mathcal{A} = \sum_n \frac{C_n^2\hbar}{M\Omega_n\omega^2}\left[I_1(n) + \left(\coth\frac{\Omega_n\hbar}{2kT} - 1\right)I_2(n)\right] \tag{36}$$

The above expression is for the most general case corresponding to independent evolution of the oscillators of the bath, each starting from a thermal equilibrium state with temperature T. If however we use the same approximation as the one used by Caldiera-Legget or Dowker and Halliwell (1992) then r.h.s. simplifies considerably since the oscillators of the bath are assumed to be in thermal equilibrium for all time; in such a case

$$I_1(n) = I_2(n) = \frac{2}{\omega^2}\sin^2\omega(t - t_0) \tag{37}$$

Following Caldeira and Legget if we choose a continuum of oscillators with density $\rho_D(\Omega)$, we simply make the replacement

$$\sum_n \to \int \mathrm{d}\Omega \rho_D(\Omega), C_n \to C(\Omega)$$

so that we finally have

$$\mathcal{A} = \frac{2\hbar}{M\omega^4} \sin^2 \omega(t - t_0) \int_0^\infty C^2(\Omega)\rho_D(\Omega)\mathrm{coth}\frac{\Omega}{2kT}\mathrm{d}\Omega \tag{38}$$

The complex representation of $\mathcal{A}$ even in its most general form (35) provides an easy interpretation of the main characteristics of the test oscillator; besides renormalization of the frequency the dissipation due to bath is apparent from (34). Next we note that (32) corresponds to the situation when the test oscillator is initially constrained at a fixed position x_0; if on the other hand we take the initial configuration to correspond to the coherent state α, then defining

$$p(x, t|\alpha, t_0) = \pi(x, t|\alpha, t_0)|_{\text{normalized}} \tag{39}$$

$$\pi(x, t|\alpha, t_0) = \int \pi_{\text{final}}(x, t|x_0, t_0) \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{m\omega}{\hbar}(x_0 - \alpha)^2\right] \mathrm{dx_0} \tag{40}$$

we find that $p(x_1, t_1|\alpha, t_0; I\!\!B)$ is given by

$$p(x_1, t_1|\alpha, t_0, I\!\!B) = \pi(x_1, t_1|\alpha, t_0, I\!\!B)|_{\text{mod}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{-\frac{m\omega}{\hbar}[x_1 - \alpha_1 \cos\omega(t_1 - t_0) - \alpha_2 \sin\omega(t_1 - t_0) - F(t_1, t_0)]^2\right\}. \tag{41}$$

Next we pass the oscillator through a slit centered at $\overline{x}$ at time t_1 to obtain

$$p(\overline{x}, t_1|\alpha, t_0, I\!\!B) = \left(\pi\{\sigma^2 + \frac{\hbar}{m\omega}\}\right)^{-1/2} \exp\left\{-\frac{[\overline{x} - F_\alpha(t_1, t_0)]^2}{\sigma^2 + \frac{\hbar}{m\omega}}\right\}. \tag{42}$$

where $F_\alpha(t_1, t_0)$ is given by

$$F_\alpha(t_1, t_0) = F(t_1 - t_0) + \alpha_1 \cos\omega(t_1 - t_0) + \alpha_2 \sin\omega(t_1 - t_0) \tag{43}$$

We next note that under the assumption that each of the bath oscillators is under thermal equilibrium,a major simplification arises namely the measure density p continues to be positive real valued even after the conditioning of the bath variables are removed by virtue of $\mathcal{A}$ remaining positive real valued(see (37)). Thus removing the conditioning,we obtain

$$p(\overline{x}, t_1|\alpha, t_0) = \left(\pi\{\sigma^2 + \frac{\hbar}{m\omega}\}\right)^{-1/2} \int (2\pi\mathcal{A})^{-1/2} \exp -\frac{z^2}{\mathcal{A}}$$

$$\exp\left\{-\frac{[\overline{x}-\alpha_1\cos\omega(t_1-t_0)-\alpha_2\sin\omega(t_1-t_0)-z]^2}{\sigma^2+\frac{\hbar}{m\omega}}\right\}\mathrm{dz}$$

$$(44)\quad =\left(\pi[2\mathcal{A}+\sigma^2+\frac{\hbar}{m\omega}]\right)^{1/2}\exp\left\{-\frac{[\overline{x}-\alpha_1\cos\omega(t_1-t_0)-\alpha_2\sin\omega(t_1-t_0)]^2}{2\mathcal{A}+\sigma^2+\frac{\hbar}{m\omega}}\right\}$$

This elegant formula is one of the significant results following from CMTF and can be used as a test of the theory itself.

It is needless to repeat the remarks made in the earlier section which are quite pertinent in the present context. The probability density function of $X(t_2)$ conditional on the oscillator having passed through the slit centered at $\overline{x}$ at time t_1 can be evaluated using the same type of analysis presented earlier; we thus have

$$(45)\qquad p(x_2,t_2|\overline{x},\sigma,t_1;\alpha,t_0,I\!B)=\left(\frac{m\omega K}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\omega K}{\hbar}[x_2-H_\alpha]^2\right\}$$

where K is defined by (11) and H_α is given by

$$(46)\qquad H_\alpha=F(t_2,t_1)+\{\overline{x}+\frac{m\omega\sigma^2}{\hbar}F_\alpha(t_1-t_0)\}G$$

where again G is defined by(12). It thus follows that the conditional probability of passage through the second slit is given by

$$(47)\qquad p(\overline{y},\rho,t_2|\overline{x},\sigma,t_1\alpha,t_0,I\!B)=\left(\pi\{\rho^2+\frac{\hbar}{m\omega K}\}\right)^{-1/2}\exp\left\{\frac{(\overline{y}-H_\alpha)^2}{\rho^2+\frac{\hbar}{m\omega K}}\right\}.$$

Again this result can be put to test after removing the conditioning if the gadenken for the Gaussian slit can be realized physically.

5 Summary and conclusions

In this contribution we have examined the problem of assignment of probabilities to the histories of paths from the point of view of Complex measure theory proposed earlier. The motivation is essentially due to the fact that deeper problems exist whenever probabilistic analysis beyond the framework of Born interpretation is resorted to. For instance when a Gaussian slit is represented by the projector, it gives rise to overlapping probabilities and the usual Born recipe doesn't work; unfortunately in the Hilbert space framework, all vectors are automatically normed to one. The Complex Measure Theoretic Framework brings out in a transparent way the anomaly. In this case we have shown that there will be an impasse if we resort to compute the complex measure of the transit through the slit(the analogue of amplitude of the passage through the slit in the conventional quantum mechanical approach). Thus the only way out is to arrive at the variational measure (which is normalized) and then use the same to compute the probability of passage through the slit. Using this method of analysis, we have shown how we can handle passage through several slits. The method of analysis is transparent and shows how the process is to be stopped and restarted,a scenario which is very close to the manner the experiments are designed. Thus in CMTF we have a natural way to deal with situations by introducing appropriate conditional measures. The probabilities of passage through the slits are very crucial quantitative measures and can be subjected to experimental test directly. It is with this motive behind we have presented the results relating to the test oscillator in a bath.

References

1 Caldeira A O and Leggett A J 1983 *Physica 121 A* 587

2 Cameron R H 1960 *J.Math. Phys. 39* 126

3 Carusotto S 1975 *Phys. Rev. A 11* 1407

4 Dirac P A M 1933 *Physikalische Zeitschrift der Sowjet Union 3*(also in Schwinger J (ed) (1958) Selected Papers in Quantum Electrodynamics, Dover, New York.

5 Dowker H F and Halliwell J J 1992 *Phys Rev D 46* 1580

7 Feynman R P 1948 *Rev. Mod. Phys. 20* 267.

8 Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals,* (New York: Mc Graw Hill)

9 Feynman R P and Vernon J R 1963 *Ann Phys (N. Y.) 24* 118

10 Ford G W, Kac M and Mazur P 1965 *J. Math Phys 6* 504

11 Gelfand I M and Yagalom A M 1956 *Uspekhi Mat naak 11* 77

12 Gellman M and Hartle J B 1993 *Phys. Rev. D 47* 3345.

13 Ito K 1961 *Weiner Integral and Feynman Integral* in Proceedings of the Fourth Berkely Symposium in Mathematical Statistics and Probability Ed. J. Neyman and E L Scott (University of California Press, Berkeley) Vol II p 227-238

14 Ito K 1967 *Generalized Uniform Complex Measures in Hilbert Metric Space* in Proceedings of the Fifth Berkely Symposium on Mathematical Statistics and Probability Ed. J. Neyman and E L Scott (University of California Press, Berkeley) Vol II Part 1 p 145-61

15 Skorobogatov G A and Svertilov S I 1998 *Phys. Rev. A 58* 3426.

16 Srinivasan S K 1997 *J Phys A Math Gen 30* 8297

17 Srinivasan S K 1998 *J Phys A Math Gen 31* 4541

18 Srinivasan S K 2001 *Quantum Phenomena via Complex Measures* in Advances in Probability and Stochastic Processes ed Krishnamoorthy A, Raju N and Ramaswami V (New Jersey: Notable

19 Srinivasan S K 2005a *Generalized Chebycheff Theorem for Complex and Quaternion Probability Measures* Proceedings of the National Seminar on Applied Stochastic Processes, Chennai Ed R.Natarajan (to be published)

20 Srinivasan S K 2005b *Quantum Phenomena via Complex Measure: Holomorphic Extension* (to be published)

21 Srinivasan S K and Sudarshan E C G 1994 *J Phys A Math Gen 27* 517

22 Von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics*(New Jersey: Princeton University Press).

23 Youseff S 1991 *Mod Phys Lett A 6* 225

24 Youseff S 1994 *Mod Phys Lett A 9* 2571