

Application of the Lie-Trotter-Kato (LTK) product formula to calculation of propagators and the generating functionals $Z(J)$ in the Quantum Mechanics and in the Quantum Field Theory. Quantum (Exotic) stochastic aspect

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Abstract. Quantum stochastic (exotic) formulae were obtained for: Propagator for nonrelativistic massive spinless particle in a potential (Section A), scalar, neutral massive field with a self-interaction (Section B), free photon field in the vacuum (Section C). These results, in turn, lead to the quantum stochastic (exotic) formulae for corresponding generating functionals $Z(J)$.

INTRODUCTION

It became clear over time that a natural place of the Feynman integrals, [1–4], is provided by the **exotic**, or **quantum stochastic processes** (in terms of the **probability amplitudes** rather than in probabilities themselves). Contributions by mathematicians notably by M. Kac [5] and others stay on a solid ground of classic probability theory able to grasp statistical aspects of the quantum theory, only. Review of attempts to justify rigorously the Feynman integrals can be found in, [6]. A novel, effective approach to the problem has been proposed by E. Nelson, [7], using the LTK product formula, [8]. The same approach but in a more pedestrian form suitable for the physicist audiences was given by L.S. Schulman, [9], who used the Euclidean time and a standard theory of stochastic processes. We give in Section A an exotic and improved version of his work using Minkowski's time by completing the proof of the basic formula for the propagator (by the mathematical induction and use of independence of quantum random variables), and by extending the scheme to scalar quantum field and also to the pure electrodynamics. The later case deals with Riemann-Silberstein vector formulation of the theory, [10]. The LTK product formula results in an explicit formulae for the quantum (exotic) stochastic processes, propagators, as well as in quantum (exotic) stochastic formulae for the generating functionals $Z(J)$, [11].

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A. MASSIVE, NON-RELATIVISTIC PARTICLE ON LINE IN A POTENTIAL $V(x)$

The Hamiltonian, in the coordinate representation, is

$$H = -\frac{1}{2m}\left(\frac{d}{dx}\right)^2 + V(x), \quad (\hbar = 1) \quad (1)$$

The Schrödinger equation with an initial condition at $t = 0$ reads

$$i\partial_t \psi_t = H\psi_t, \quad t \geq 0, \quad \psi_{t=0} \equiv \psi_0 - \text{given state-vector} \quad (2)$$

Hence, for t positive

$$\psi_t \equiv U_t \psi_0 = \exp(-itH)\psi_0, \quad (3)$$

In the coordinate representation, we get for the wave function

$$\psi(x, t) = \langle x | \psi_t \rangle = \langle x | U_t | \psi_0 \rangle = \int_{\mathbb{R}} dy \langle x | U_t | y \rangle \langle y | \psi_0 \rangle \equiv \int_{\mathbb{R}} dy K(x, t; y, 0) \psi_0(y) \quad (4)$$

Due to the stationarity of the potential, the propagator K depends on times via their difference, only

$$K(x, t; y, 0) \equiv \langle x | U_t | y \rangle = K(x, t''; y, t'), \quad t \equiv t'' - t' \geq 0 \quad (5)$$

Using the LTK product formula, we get

$$\exp[t(A + B)] = \lim_{n \rightarrow \infty} [\exp(\varepsilon A) \exp(\varepsilon B)]^n = \lim_{n \rightarrow \infty} [\exp(\varepsilon B) \exp(\varepsilon A)]^n, \quad (6)$$

$$\varepsilon \equiv \frac{t}{n} = \varepsilon(n)$$

where, in our case

$$A \equiv \frac{i}{2m} \left(\frac{d}{dx}\right)^2, \quad B \equiv -iV(x) \quad (7)$$

we get for the evolution operator U_t in the coordinate representation

$$U_t = \lim_{n \rightarrow \infty} \left\{ \exp[-i\varepsilon V(x)] \exp\left[i\varepsilon \frac{1}{2m} \left(\frac{d}{dx}\right)^2\right] \right\}^n \quad (8)$$

The second-order derivative in the exponent can be reduced to the first-order using the well known Fresnel integral formula [13]

$$\left(\frac{a}{2\pi i}\right)^{1/2} \int_{\mathbb{R}} dy \exp(i\frac{a}{2}y^2 \pm iby) \equiv E[\exp(\pm ib\hat{y})] = \exp(-i\frac{b^2}{2a}), \quad a > 0 \quad (9)$$

Corresponding formula for negative a follows from here by the complex conjugation. Assuming

$$a = m, \quad b = i\sqrt{\varepsilon} \frac{d}{dx} \quad (10)$$

we obtain the following parametric representation

$$\begin{aligned} \exp[i\varepsilon \frac{1}{2m} \left(\frac{d}{dx}\right)^2] &= \\ &= \left(\frac{m}{2\pi i}\right)^{1/2} \int_{\mathbb{R}} dy \exp(i\frac{m}{2}y^2 + \sqrt{\varepsilon} y \frac{d}{dx}) \equiv E[\exp(\sqrt{\varepsilon} \hat{y} \frac{d}{dx})] \end{aligned} \quad (11)$$

Here \hat{y} stands for a quantum random variable that is distributed with the probability amplitude, denoted by the hat, rather than with a probability itself

$$\hat{P}(y \leq \hat{y} < y + dy) \equiv \left(\frac{m}{2\pi i}\right)^{1/2} \exp(i\frac{m}{2}y^2) dy \quad (12)$$

Generally, as it is well known, $|\hat{P}(A)|^2 = P(A)$ where A is some event and $P(A)$ is its probability. The coefficient in front of the integral in (11) ensures the conventional normalization

$$\hat{E}(1) = 1 \quad (13)$$

Strictly speaking one should have placed the hat on top of E in the formula (11) but the very presence of \hat{y} reminds us that we are in the quantum domain. Anyway, we omit the hats whenever it does not leads to a confusion.

It will be convenient for what follows to write the formula (8) as a product of equal factors containing independent and identically distributed random variables \hat{y}_k . This in turn allows us to write the product of expected values as a single expectation value of the factors, viz.,

$$\begin{aligned} U_t &= \lim_{n \rightarrow \infty} (E\{\exp[-iV(x)] \exp(\sqrt{\varepsilon} \hat{y}_n \frac{d}{dx})\} \dots E\{\exp[-iV(x)] \exp(\sqrt{\varepsilon} \hat{y}_1)\}) = \\ &= \lim_{n \rightarrow \infty} E\{\exp[-i\varepsilon V(x)] \exp(\sqrt{\varepsilon} \hat{y}_n \frac{d}{dx}) \dots \exp[-i\varepsilon V(x)] \exp(\sqrt{\varepsilon} \hat{y}_1 \frac{d}{dx})\} \end{aligned} \quad (14)$$

The ordering of factors is immaterial as it amounts to a different numbering of the variables \hat{y}_k which can be arbitrary. In the process we have inserted factors with $V(x)$ under the expectation signs which are benign operations. The remaining task consists of transporting the derivatives to the right using the well known identity

$$\exp(u \frac{d}{dx}) \exp[F(x)] = \exp[F(x + u)] \exp(u \frac{d}{dx}) \quad (15)$$

In our case

$$F(x) \equiv -i\varepsilon V(x)$$

$$u_k \equiv \sqrt{\varepsilon} \hat{y}_k, \quad k = 1, 2, \dots, n \quad (16)$$

Consider first the simplest cases with $n = 1, 2, 3$ and before the averaging

$$\begin{aligned} & \exp[F(x)] \exp(u_3 \frac{d}{dx}) \exp[F(x)] \exp(u_2 \frac{d}{dx}) \exp[F(x)] \exp(u_1 \frac{d}{dx}) = \\ & = \exp[F(x)] \exp(u_3 \frac{d}{dx}) \exp[F(x) + F(x + u_2)] \exp(u_1 + u_2) \frac{d}{dx} = \\ & = \exp[F(x) + F(x + u_3) + F(x + u_2 + u_3)] \exp(u_1 + u_2 + u_3) \frac{d}{dx} \end{aligned} \quad (17)$$

The identity is valid for an arbitrary coefficient ε . Missing of the u 's with lower numbers in the arguments of F suggests that we are dealing with differences of two sums. This suggests the following conjecture, [9]

$$U_t = \lim_{n \rightarrow \infty} E \{ \exp[-i\varepsilon \sum_{k=1}^n V(x + \hat{b}_k - \hat{b}_n)] \exp(-\hat{b}_n \frac{d}{dx}) \} \quad (18)$$

where we have denoted

$$\hat{b}_k \equiv -\sqrt{\varepsilon} \sum_{l=1}^k \hat{y}_l, \quad k = 0, 1, 2, \dots, n, \quad (\hat{b}_0 \equiv 0) \quad (19)$$

This formula may be proven by the mathematical induction with respect to n . Namely, assuming its validity for some n and an arbitrary constant ε we consider the next product of the $n + 1$ factors which can be presented as a product of two averages: the new factor followed by the expectation value of the previous n factors with $\varepsilon(n)$ replaced by $\varepsilon(n + 1)$. The factorization results from the independence of the new random variable \hat{y}_{n+1} and all the other variables with smaller numbers entering the second average. Second average may be written in the form (18) by the assumption. The average on the left is

$$E \{ \exp[-i\varepsilon(n + 1)V(x)] \exp(u_{n+1} \frac{d}{dx}) \}$$

with the new quantum (exotic) random variable, \hat{y}_{n+1}

$$u_{n+1} = \sqrt{\varepsilon(n + 1)} \hat{y}_{n+1}$$

that is independent from all other with smaller subindices from the second average. This again allows to write the product of two expectation values as a single expectation of multipliers. Applying now the formula (15) we get argument x shifted by the u_{n+1} . The same is true for new coefficient at $\frac{d}{dx}$ under the exponential sign. However, the

emerging combination, according to (19) equals

$$u_{n+1} - \hat{b}_n = -\hat{b}_{n+1}$$

which effectively increases the upper limit of summation to $n + 1$, and replaces the coefficient in front of the derivative to required value of $-\hat{b}_{n+1}$. The $\varepsilon(n)$ was initially replaced by $\varepsilon(n + 1)$. Thus the formula (17) holds for $n + 1$ factors. Finally, the formula (18) holds for $n = 1$ as it follows from (15), by taking its expectation value E . This proves the formula (18) for an arbitrary n .

We introduce now an interpolating function $\hat{b}(s) \equiv \hat{b}_s$ coinciding with \hat{b}_k for $s = k\varepsilon$

$$\hat{b}(k\varepsilon) = \hat{b}_k \quad k = 0, 1, 2, \dots, n \quad (20)$$

$$\hat{b}_0 \equiv 0, \quad \hat{b}(n\varepsilon) = \hat{b}_t$$

We find for the averages of \hat{b}'_s using (9) and the fact that the random variables \hat{y}_p are mutually independent and identically distributed

$$\begin{aligned} E(\hat{b}_j) &= \sqrt{\varepsilon} \sum_{p=1}^j E(\hat{y}_p) = j\sqrt{\varepsilon} E(\hat{y}_1) = j\sqrt{\varepsilon} \left(\frac{m}{2\pi i}\right)^{1/2} \int_{\mathbb{R}} dy \exp(i\frac{m}{2}y^2) y = \\ &= j\sqrt{\varepsilon} \frac{d}{db} \exp(-i\frac{b^2}{2m})|_{b=0} = 0 \quad j = 0, 1, 2, \dots, n \end{aligned} \quad (21)$$

$$E(\hat{b}_j \hat{b}_k) = \varepsilon \sum_{q=1}^j \sum_{p=1}^k E(\hat{y}_p \hat{y}_q) = \varepsilon \sum_p^{j \wedge k} E(\hat{y}_p^2) = \varepsilon j \wedge k E(\hat{y}_1^2) = i(j \wedge k) \frac{\varepsilon}{m} \quad (22)$$

This comes about because only terms with $q = p$ contribute due to the independence of the random variables and vanishing of their averages. Further, their identical distributions leads to equal expectations of their squares. The above properties are characteristic for the quantum (exotic) Brownian motion variables. In the continuum limit when $\varepsilon \rightarrow 0$ we end up with the quantum (exotic) Brownian motion process. The last quantum (exotic) expectation in (22) is found in a similar way

$$E(\hat{y}_1^2) = \left(\frac{m}{2\pi i}\right)^{1/2} \int_{\mathbb{R}} dy \exp(i\frac{m}{2}y^2) y^2 = -\left(\frac{d}{db}\right)^2 \exp(-i\frac{b^2}{2m})|_{b=0} = \frac{i}{m} \quad (23)$$

which may be written in more clear, physical way

$$\Delta \hat{b}_t \equiv (E[(\Delta \hat{b}_t)^2])^{1/2} = (i\Delta t)^{1/2}, \quad \Delta t = \varepsilon \quad (24)$$

The quantum (exotic) Brownian motion possesses **continuous trajectories that are non-differentiable**.

Acting with U_t on the initial wave function $\psi_0(x)$, we obtain

$$\begin{aligned}
\psi(x, t) &= \lim_{n \rightarrow \infty} E\{\exp[-i\varepsilon \sum_{k=1}^n V(x + \hat{b}_k - \hat{b}_n)] \psi_0(x - \hat{b}_n)\} = \\
&= E\{\exp[-i \int_0^t ds V(x + \hat{b}_s - \hat{b}_t)] \psi_0(x - \hat{b}_t)\} = \\
&= \int_{\mathbb{R}} dy E\{\exp[-i \int_0^t ds V(y + \hat{b}_s)] \delta(x - y - \hat{b}_t)\} \psi_0(y) \quad (25)
\end{aligned}$$

According to the formula (4), we get for the propagator

$$K(x, t; y, 0) = E\{\exp[-i \int_0^t ds V(y + \hat{b}_s)] \delta(x - y - \hat{b}_t)\} = \quad (26)$$

$$= \lim_{n \rightarrow \infty} E\{\exp[-i\varepsilon \sum_{k=1}^n V(y + \hat{b}_k)] \delta(x - y - \hat{b}_n)\} \quad (27)$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} dy_1 \dots \int_{\mathbb{R}} dy_n \left(\frac{m}{2\pi i}\right)^{n/2} \exp[i\frac{m}{2} \sum_{k=1}^n y_k^2 - i\varepsilon \sum_{k=1}^n V(y - \sqrt{\varepsilon} \sum_{l=1}^k y_l)] \times \\
&\quad \times \delta(x - y + \sqrt{\varepsilon} \sum_{l=1}^n y_l) \quad (28)
\end{aligned}$$

The formula (26) is the quantum counterpart of the Feynman–Kac formula. To elucidate this point, we change the integration variables

$$\begin{aligned}
z_0 &= y \\
z_1 &= y - \sqrt{\varepsilon} y_1 & y_1 &= -\frac{1}{\sqrt{\varepsilon}}(z_1 - z_0) \\
z_2 &= y - \sqrt{\varepsilon}(y_1 + y_2) & y_2 &= -\frac{1}{\sqrt{\varepsilon}}(z_2 - z_1) \\
.. & & & \\
..... & & & \\
z_n &= y - \sqrt{\varepsilon}(y_1 + y_2 + \dots + y_n) & y_n &= -\frac{1}{\sqrt{\varepsilon}}(z_n - z_{n-1})
\end{aligned} \quad (29)$$

The modulus of the Jacobian is

$$\left| \frac{\partial(y_1, \dots, y_n)}{\partial(z_1, \dots, z_n)} \right| = (\sqrt{\varepsilon})^{-n} \quad (30)$$

which allows to rewrite the propagator, for t positive, as follows

$$\begin{aligned}
&K(x, t; y, 0) = \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{dz_1}{M_\varepsilon} \dots \int_{\mathbb{R}} \frac{dz_n}{M_\varepsilon} \exp\{i[\frac{m}{2} \varepsilon \sum_{k=1}^n [(z_k - z_{k-1})^2 \varepsilon^{-2} - V(z_k)]]\} \delta(x - z_n) \quad (31)
\end{aligned}$$

where we denoted

$$M_\varepsilon \equiv \left(\frac{2\pi i \varepsilon}{m}\right)^{1/2} \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0 \quad (32)$$

Let $z(s)$ be an interpolating function such that

$$z(k\varepsilon) = z_k = y + b_k \quad k = 0, 1, 2, \dots, n \quad (33)$$

then, in the limit of a continuous time we get from (31)

$$K(x, t ; y, 0) = \prod'_{s=0} \int_{\mathbb{R}} \frac{dz_s}{M_0} \exp[i \int_0^t ds L(z_s, \dot{z}_s)]_{|z_0=y, z_t=x} \quad (34)$$

which is customarily written as a single Feynman integral over continuous paths connecting end-points y and x , and where the phase is determined by the Lagrangian

$$L(z, \dot{z}) \equiv \frac{m}{2} (\dot{z})^2 - V(z) \quad (35)$$

Notice that the exotic stochastic formula (26) is free of the embarrassing $M_0 = 0$ in the denominators. The path's continuity is guaranteed for massive particles, only. Massless photon's path has **cadlag** property, [11]. Changing the variable from s to u

$$s = u + t/2, \quad -t/2 \leq u \leq t/2, \quad z_s = \tilde{z}_u \quad (36)$$

we get for K omitting tildas over z since both set of functions are equally good variables of the functional integration, and renaming back $u \rightarrow s$ we get similar foemula for K with just the boundary values changed

$$K(x, t ; y, 0) = \int \prod_{s=-t/2}^{t/2} \frac{dz_s}{M_0} \exp[i \int_{-t/2}^{t/2} ds L(z_s, \dot{z}_s)]_{|z(-t/2)=y, z(t/2)=x} = K(x, t/2 ; y, -t/2) \quad (37)$$

THE $Z(J)$ GENERATING FUNCTIONAL

Using (26) we consider now the conditional amplitude

$$K(0, \infty ; 0, -\infty) \equiv \lim_{t \rightarrow \infty} K(0, t/2 ; 0, -t/2) = \lim_{t \rightarrow \infty} E\{\exp[-i \int_{-t/2}^{t/2} ds V(\hat{b}_{s+t/2})] \delta(\hat{b}_t)\} \quad (38)$$

and a similar amplitude under presence of an external non-random source J

$$K_J(0, \infty ; 0, -\infty) \equiv \lim_{t \rightarrow \infty} E\{\exp[-i \int_{-t/2}^{t/2} ds [V(\hat{b}_{s+t/2}) - J_{s+t/2} \hat{b}_{s+t/2}] \delta(\hat{b}_t)]\} =$$

$$= \prod_{s=-\infty}^{\infty} \int_{\mathbb{R}} \frac{dz_s}{M_0} \exp \left\{ i \int_{\mathbb{R}} ds [L(z_s, \dot{z}_s) + J_s z_s] \right\} \Big|_{z(-\infty)=z(\infty)=0} \quad (39)$$

The ratio of the two amplitudes is a familiar generating functional

$$Z(J) \equiv \frac{K_J(0, \infty; 0, -\infty)}{K(0, \infty; 0, -\infty)}, \quad Z(0) = 1 \quad (40)$$

which may be expressed using Feynman integrals with M_0 factors removed or by the quantum (exotic) stochastic formulae using (38), (39).

APPLICATION

We would like to calculate the free propagator corresponding to $V = 0$

$$K_0(x, t; y, 0) \equiv E\{\delta(x - y - \hat{b}_t)\} = \lim_{n \rightarrow \infty} E\{\delta(x - y - \hat{b}_n)\} \quad (t = n\varepsilon) \quad (41)$$

We first use the integral representation of the δ -function

$$K_0(x, t; y, 0) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda E\{\exp[i\lambda(x - y - \hat{b}_n)]\} \quad (42)$$

Using the definition of \hat{b}_n given in (19) we get

$$K_0(x, t; y, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda e^{i\lambda(x-y)} \lim_{n \rightarrow \infty} E\{\exp[i\lambda\sqrt{\varepsilon} \sum_{k=1}^n \hat{y}_k]\} \quad (43)$$

where \hat{y}_k are independent and identically distributed quantum (exotic) random variables. Hence the average is a product of individual averages that are equal to one another. As a result, we get

$$K_0(x, t; y, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda e^{i\lambda(x-y)} \lim_{n \rightarrow \infty} E\{\exp(i\lambda\sqrt{\varepsilon}) \hat{y}_1\}^n \quad (44)$$

Using the definition (9) of the quantum (exotic) average, we get

$$K_0(x, t; y, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda e^{i\lambda(x-y)} \lim_{n \rightarrow \infty} \left\{ \left(\frac{m}{2\pi i} \right)^{1/2} \int_{\mathbb{R}} dy_1 \exp\left(i\frac{m}{2} y_1^2 + i\lambda\sqrt{\varepsilon} y_1\right) \right\}^n \quad (45)$$

Performing Fresnel integral using (9) again, we get for $t > 0$

$$K_0(x, t; y, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} d\lambda \exp\left[-i\frac{t}{2m} \lambda^2 + i(x-y)\lambda\right] = \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left[i\frac{m}{2t}(x-y)^2\right] \quad (46)$$

the last step entailed another Fresnel integral (in its complex conjugated form). The

limit operation is rather trivial since it amounts to a replacement of $n\varepsilon$ with t which is fixed and the dependence on n disappears. In some other system of units when \hbar enters the calculations explicitly we end up with the same formula with t replaced by $\hbar t$, on the right hand side of (46).

B. MASSIVE, SCALAR AND NEUTRAL FIELD

We are interested in a dynamics that corresponds to the Lagrangian density

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi), \quad m \neq 0 \quad (1)$$

where $\phi(\mathbf{x})$ is a field operator canonically conjugated to a momentum operator

$$\Pi(\mathbf{x}) \equiv \frac{\partial L}{\partial \dot{\phi}(\mathbf{x})} = \dot{\phi}(\mathbf{x}) \quad (2)$$

They do satisfy the canonical, equal-time, commutation relations. In the coordinate representation the $\phi(\mathbf{x})$ acts as a multiplication operator while the canonical momentum is represented by the functional derivative

$$\Pi(\mathbf{x}) = -i \frac{\delta}{\delta \phi(\mathbf{x})} \quad (3)$$

The Hamiltonian corresponding to L is given by the formula

$$\begin{aligned} H &= \int_{\mathbb{R}^3} d\mathbf{x} \left\{ \frac{1}{2} [\Pi^2(\mathbf{x}) + \sum_{k=1}^3 [\partial_k \phi(\mathbf{x})]^2 + m^2 \phi^2(\mathbf{x})] + V[\phi(\mathbf{x})] \right\} = \\ &\equiv -\frac{1}{2} \left(\frac{\delta}{\delta \phi} \right)^2 + \tilde{V}(\phi) \end{aligned} \quad (4)$$

where

$$\tilde{V}(\phi) \equiv V(\phi) + \frac{1}{2} \sum_{k=1}^3 (\partial_k \phi)^2 + \frac{1}{2} m^2 \phi^2 \quad (5)$$

Also, we simultaneously omit the integration sign and the argument \mathbf{x} .

Examples

$$\begin{aligned} \phi^2 &\equiv \int_{\mathbb{R}^3} d\mathbf{x} \phi^2(\mathbf{x}) \\ V(\phi) &\equiv \int_{\mathbb{R}^3} d\mathbf{x} V[\phi(\mathbf{x})], \quad \text{etc.} \end{aligned} \quad (6)$$

This increases transparency and facilitates the treatment of the second-quantized case in a way that is similar to the previously considered first-quantized problem. The evolution operator in this notation is given by

$$U_t = \exp(-itH) = \exp\{it[\frac{1}{2}(\frac{d}{d\phi})^2 - \tilde{V}(\phi)]\} \quad (7)$$

According to the LTK product formula it may be written as the limit

$$U_t = \lim_{n \rightarrow \infty} (\exp[-i\varepsilon \tilde{V}(\phi)] \exp[\frac{i\varepsilon}{2}(\frac{\delta}{\delta\phi})^2])^n, \quad \varepsilon \equiv \frac{t}{n} \quad (8)$$

Using the following parametric representation

$$\frac{1}{N} \int \prod_x d\eta(\mathbf{x}) \exp(\frac{i}{2}\eta^2 + \xi\eta) \equiv E\{\exp(\xi\hat{\eta})\} = \exp(\frac{i}{2}\xi^2) \quad (9)$$

where the normalization factor N ensures that

$$E\{1\} = 1 \quad (10)$$

we get by substituting $\xi = \sqrt{\varepsilon} \frac{\delta}{\delta\phi}$

$$\exp[\frac{i\varepsilon}{2}(\frac{\delta}{\delta\phi})^2] = \frac{1}{N} \int \prod_x d\eta(\mathbf{x}) \exp(\frac{i}{2}\eta^2 + \sqrt{\varepsilon} \eta \frac{\delta}{\delta\phi}) \equiv E\{\exp(\sqrt{\varepsilon} \hat{\eta} \frac{\delta}{\delta\phi})\} \quad (11)$$

The functional integral over $\eta(\mathbf{x})$ is calculated by shifting $\eta \rightarrow \eta + \zeta$ and adjusting the fixed function ζ so that the term linear in η vanishes. The hat over η in the formulae (9) and (11) signifies the fact that it is a quantum (exotic) random field distributed with the probability amplitude

$$\frac{1}{N} \exp(\frac{i}{2}\eta^2) \prod_x d\eta(\mathbf{x}) \quad (12)$$

Upon substitution of (11) into the formula (8) we get for the evolution operator

$$U_t = \lim_{n \rightarrow \infty} \prod_{k=1}^n E\{\exp[-i\varepsilon \tilde{V}(\phi)] \exp(\sqrt{\varepsilon} \hat{\eta}_k \frac{\delta}{\delta\phi})\} \quad (13)$$

We started labelling the random fields although they are assumed independent and identically distributed which leads to equal averages. An order of equal multipliers is immaterial and amounts to some re-labelling of the quantum random fields. Because of their independence we may write the product of average values as a single average of the multipliers

$$U_t = \lim_{n \rightarrow \infty} E \left\{ \prod_{k=n}^1 \exp[-i\varepsilon \tilde{V}(\phi)] \exp(\sqrt{\varepsilon} \hat{\eta}_k \frac{\delta}{\delta \phi}) \right\} \quad (14)$$

For transporting the derivatives to the right we use the identity

$$\exp(u \frac{\delta}{\delta \phi}) \exp[F(\phi)] = \exp[F(\phi + u)] \exp(u \frac{\delta}{\delta \phi}) \quad (15)$$

with the identifications

$$\begin{aligned} F(\phi) &= -i\varepsilon \tilde{V}(\phi) \\ u_k &= \sqrt{\varepsilon} \hat{\eta}_k, \quad k = 1, 2, \dots, n \end{aligned} \quad (16)$$

In analogy to (A18), we obtain for U_t

$$U_t = \lim_{n \rightarrow \infty} E \left\{ \exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\phi + \hat{B}_k - \hat{B}_n)] \exp(-\hat{B}_n \frac{\delta}{\delta \phi}) \right\} \quad (17)$$

where we denoted the quantum (exotic) random fields

$$\hat{B}_k(\mathbf{x}) \equiv -\sqrt{\varepsilon} \sum_{l=1}^k \hat{\eta}_l(\mathbf{x}), \quad k = 0, 1, 2, \dots, n \quad (18)$$

$$\hat{B}_0(\mathbf{x}) \equiv 0 \quad (19)$$

Using (9) we may find the quantum averages

$$E[\hat{B}_k(\mathbf{x})] = 0, \quad k = 0, 1, 2, \dots, n \quad (20)$$

$$E\{\hat{B}_j(\mathbf{x}) \hat{B}_k(\mathbf{y})\} = \varepsilon \sum_{l=1}^{j \wedge k} E\{\hat{\eta}_l(\mathbf{x}) \hat{\eta}_l(\mathbf{y})\} = i\varepsilon j \wedge k \delta(\mathbf{x} - \mathbf{y}) \quad (21)$$

We also consider an interpolating quantum (exotic) random field coinciding with \hat{B}_k at $s = k\varepsilon$

$$\hat{B}(\mathbf{x}, k\varepsilon) = \hat{B}_k(\mathbf{x}), \quad k = 1, 2, \dots, n \quad (22)$$

Corresponding relation to (21), in the limit of $\varepsilon \rightarrow 0$ reads

$$E\{\hat{B}(\mathbf{x}, r) \hat{B}(\mathbf{y}, s)\} = i r \wedge s \delta(\mathbf{x} - \mathbf{y}), \quad 0 \leq r, s \leq t \quad (23)$$

$$r \wedge s \equiv \text{smaller of the numbers } r, s$$

It characterizes the quantum (exotic) Brownian random field.

Acting with the evolution operator on initial wave functional $\psi(\phi, 0)$, which may be inserted under the E -sign, we obtain the wave functional at later time $t \geq 0$

$$\begin{aligned}
\psi(\phi, t) &= \lim_{n \rightarrow \infty} E\{\exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\phi + \hat{B}_k - \hat{B}_n)] \psi(\phi - \hat{B}_n)\} = \\
&= \int \prod_{\mathbf{x}} d\phi'(\mathbf{x}) \lim_{n \rightarrow \infty} E\{\exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\phi' + \hat{B}_k)] \delta(\phi - \phi' - \hat{B}_n)\} \psi(\phi', 0) = \\
&\equiv \int \prod_{\mathbf{x}} d\phi'(\mathbf{x}) K(\phi, t; \phi', 0) \psi(\phi', 0)
\end{aligned} \tag{24}$$

where we have used the propagator for $t \geq 0$

$$K(\phi, t; \phi', 0) \equiv E\{\exp[-i \int_0^t ds \tilde{V}(\phi' + \hat{B}_s)] \delta(\phi - \phi' - \hat{B}_t)\} \tag{25}$$

which is a limit form of the prelimit expression given above. Using (9) we may rewrite the prelimit formula as follows

$$\begin{aligned}
K(\phi, t; \phi', 0) &= \lim_{n \rightarrow \infty} \frac{1}{N} \int \prod_{\mathbf{x}} d\eta_1(\mathbf{x}) \dots \frac{1}{N} \int \prod_{\mathbf{x}} d\eta_n(\mathbf{x}) \exp(\sum_{k=1}^n \frac{1}{2} \eta_k^2) \times \\
&\times \exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\phi' - \sqrt{\varepsilon} \sum_{l=1}^k \eta_l)] \delta(\phi - \phi' + \sqrt{\varepsilon} \sum_{l=1}^n \eta_l)
\end{aligned} \tag{26}$$

We change the variables of integration as follows

$$\begin{aligned}
\phi_0 &= \phi' \\
\phi_1 &= \phi' - \sqrt{\varepsilon} \eta_1 & -\eta_1 &= \frac{1}{\sqrt{\varepsilon}} (\phi_1 - \phi_0) \\
\phi_2 &= \phi' - \sqrt{\varepsilon} (\eta_1 + \eta_2) & -\eta_2 &= \frac{1}{\sqrt{\varepsilon}} (\phi_2 - \phi_1) \\
&\dots\dots\dots & \dots\dots\dots & \\
\phi_n &= \phi' - \sqrt{\varepsilon} (\eta_1 + \eta_2 + \dots + \eta_n) & -\eta_n &= \frac{1}{\sqrt{\varepsilon}} (\phi_n - \phi_{n-1})
\end{aligned} \tag{27}$$

Modulus of the Jacobian is

$$|\frac{\partial(\eta_1, \dots, \eta_n)}{\partial(\phi_1, \dots, \phi_n)}| = (\sqrt{\varepsilon})^{-n} \tag{28}$$

Hence we find from (26) an interpolating function; $\phi(\mathbf{x}, k\varepsilon) = \phi_k(\mathbf{x})$

$$\begin{aligned}
K(\phi, t; \phi', 0) &= \lim_{n \rightarrow \infty} \frac{1}{N} \int \prod_{\mathbf{x}} \frac{d\phi_1}{\sqrt{\varepsilon}} \dots \frac{1}{N} \int \prod_{\mathbf{x}} \frac{d\phi_n}{\sqrt{\varepsilon}} \times \\
&\times \exp\{i\varepsilon \sum_{k=1}^n [\frac{1}{2} (\phi_{k+1} - \phi_k)^2 \varepsilon^{-2} - \tilde{V}(\phi_k)]\} \delta(\phi - \phi_n) =
\end{aligned}$$

$$= \frac{1}{N} \int \prod_{\mathbf{x}, s=0}^t \frac{d\phi(\mathbf{x}, s)}{0} \exp\{i \int d\mathbf{x} \int_0^t ds L[\phi(\mathbf{x}, s), \partial_\mu \phi(\mathbf{x}, s)]\}_{|\phi(0) = \phi', \phi(t) = \phi} \quad (29)$$

where the Lagrangian is given in (1). Notice, please, that the quantum (exotic) stochastic expressions like (24), (25) are free of the apparent zeros in the denominators.

THE $Z(J)$ GENERATING FUNCTIONAL

Applying the same steps as in the Section A, we arrive at the formulae

$$\begin{aligned} K(\phi, t; \phi', 0) &= K(\phi, t/2; \phi', -t/2) = \\ &= E\{\exp[-i \int_{-t/2}^{t/2} ds \tilde{V}(\phi' + \hat{B})] \delta(\phi - \phi' - \hat{B}_t)\} \end{aligned} \quad (30)$$

Similarly, we find

$$\begin{aligned} K(0, \infty; 0, -\infty) &= \lim_{t \rightarrow \infty} K(0, t/2; 0, -t/2) = \\ &= \lim_{t \rightarrow \infty} E\{\exp[-i \int_{-t/2}^{t/2} ds \tilde{V}(\hat{B}_{s+t/2})] \delta(B_t)\} \end{aligned} \quad (31)$$

Replacing $\tilde{V}(\hat{B}_\sigma)$ by $\tilde{V}(\tilde{B}_\sigma) - J_\sigma \hat{B}_\sigma$ in the last formula we get $K_J(0, \infty; 0, -\infty)$. Hence the previous quantity, $K(0, \infty; 0, -\infty)$ may be considered as a value of K_J at $J = 0$. Their ratio determines the generating functional $Z(J)$

$$\begin{aligned} Z(J) &\equiv \frac{K_J(0, \infty; 0, -\infty)}{K(0, \infty; 0, -\infty)} = \frac{1}{N} \lim_{t \rightarrow \infty} E\{\exp\{-i \int_{-t/2}^{t/2} ds [\tilde{V}(\hat{B}_{s+t/2}) - J_{s+t/2} \hat{B}_{s+t/2}]\} \delta(\hat{B}_t)\} = \\ &= \frac{1}{N} \int \prod_x \frac{d\phi(x)}{0} \exp\{i \int dx [L(\phi, \partial_\mu \phi)(x) + J(x)\phi(x)]\} \end{aligned} \quad (32)$$

where the integration goes over all fields vanishing at \pm time infinity. The normalisation factor N is such that

$$Z(0) = 1 \quad (33)$$

The apparent zero in the measure is usually included into the normalization factor.

C. THE PURE QED IN THE VACUUM IN TERMS OF RIEMANN–SILBERSTEIN VECTOR

We shall write the Riemann–Silberstein Vector $\mathbf{f}_{RS} = \mathbf{E} + i\mathbf{B}$ without the "RS" subscript for the sake of greater transparency. For the pure QED in a vacuum the Maxwell equations read, [10]

$$\partial_t \mathbf{f} = i\nabla \times \mathbf{f} \quad (1)$$

$$\nabla \cdot \mathbf{f} = 0 \quad (2)$$

Hence \mathbf{f} , \mathbf{f}^+ without the traditional hats are field operators. The hats will be saved for the corresponding quantum (exotic) random fields instead.

Taking the time derivative of \mathbf{f} and using the double–curl identity, we find

$$\square \mathbf{f} = (\partial_t^2 - \Delta) \mathbf{f} = 0 \quad (3)$$

The equations follow from the Lagrangian

$$L = \partial_\mu \mathbf{f}^+ \cdot \partial^\mu \mathbf{f} \quad (4)$$

In order to satisfy the transversality condition (2), we assume that \mathbf{f} is the curl of some vector \mathbf{g}

$$\mathbf{f} = \nabla \times \mathbf{g} \quad (5)$$

In the components it reads

$$f^r = \epsilon^{rsu} \partial_s g^u \quad r, s, u = 1, 2, 3 \quad (6)$$

Since the potential \mathbf{g} may be gaged by adding the gradient of a scalar field Λ one may use this freedom in order to impose transversality condition on \mathbf{g}

$$\nabla \cdot \mathbf{g}(\mathbf{x}, s) = 0, \quad 0 \leq s \leq t \quad (7)$$

The same holds true for the Hermitian conjugate fields \mathbf{f}^+ and \mathbf{g}^+ .

Using (6) and the identity

$$\sum_{r=1}^3 \epsilon^{rsu} \epsilon^{rqp} = \delta^{sq} \delta^{up} - \delta^{sp} \delta^{uq} \quad (8)$$

one may express the Lagrangian in terms of the new potential

$$L = -\partial_\mu \mathbf{g}^+ \cdot \Delta \partial^\mu \mathbf{g} = -(\dot{g}^+)^r \Delta \dot{g}^r - (\Delta g^+)^r \Delta g^r \quad (9)$$

We have omitted here full divergencies assuming that they do not contribute to the action functional due to vanishing of the potentials at the spatial infinity. One notice that the generalized velocities can be determined only up to an arbitrary harmonic function. Hence, the Lagrangian is a singular one.

$$\pi_g^r \equiv \frac{\partial L}{\partial \dot{g}^r} = -\Delta(\dot{g}^+)^r \quad \pi_{g^+}^r \equiv \frac{\partial L}{\partial \dot{g}^+} = -\Delta \dot{g}^r \quad (10)$$

Equal time, non-vanishing commutation relations, compatible with the transversality of \mathbf{g} , \mathbf{g}^+ and π, π^+ are of the form

$$\begin{aligned} [g^r(\mathbf{x}), \pi_g^s(\mathbf{y})] &= iP^{rs} \delta(\mathbf{x} - \mathbf{y}) \\ [(g^+)^r(\mathbf{x}), (\pi_{g^+}^+)^s] &= iP^{rs} \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (11)$$

where P^{rs} is a projection operator on the transversal directions

$$P^{rs} \equiv \delta^{rs} - \Delta^{-1} \partial^r \partial^s \quad (12)$$

In the coordinate representation where \mathbf{g}, \mathbf{g}^+ are diagonal, we have

$$\pi_g^r = -iP^{rs} \frac{\delta}{\delta g^s}, \quad \pi_{g^+}^r = -iP^{rs} \frac{\delta}{\delta (g^+)^s} \quad (13)$$

The Hamiltonian, in this representation, is given by the expression

$$\begin{aligned} H &= \int_{\mathbb{R}^3} d\mathbf{x} [-\pi_g^r(\mathbf{x}) \Delta^{-1} \pi_{g^+}^r(\mathbf{x}) + \Delta (g^+)^r(\mathbf{x}) \Delta g^r(\mathbf{x})] \equiv \\ &\equiv \frac{\delta}{\delta (g^+)^r} \Delta^{-1} P^{rs} \frac{\delta}{\delta g^s} + \Delta (g^+)^r \Delta g^r \end{aligned} \quad (14)$$

We have used here our familiar convention suppressing both the integration sign and the variable of integration, cf. (B6). Using the LTK product formula, we find

$$U_t = \exp(-itH) = \lim_{n \rightarrow \infty} [\exp(-i\varepsilon \tilde{V}) \exp(-i\varepsilon \frac{\delta}{\delta (g^+)^r} \Delta^{-1} P^{rs} \frac{\delta}{\delta g^s})]^n \quad (15)$$

$$\tilde{V}(\mathbf{g}, \mathbf{g}^+) \equiv \Delta (g^+)^r \Delta g^r = -(\mathbf{f}^+)^r \Delta \mathbf{f}^r \equiv V(\mathbf{f}, \mathbf{f}^+) \quad (16)$$

$$\varepsilon \equiv \frac{t}{n} \quad (17)$$

The place where the Hermitian conjugation matters are the commutation relations (11), The second line is obtained from the first by the Hermitian conjugation that reverses an order of two operators what produces a minus sign in front of the commutator needed for its consistency with the first line.

We may reduce the degree of functional derivative in (15) by using the following integral identity in which $\eta^r, (\eta^+)^r$ are independent variables

$$\begin{aligned} \frac{1}{N} \int \prod_{\mathbf{x}, r} d\eta^r(\mathbf{x}) d(\eta^+)^r(\mathbf{x}) \exp[-i(\eta^+)^r P^{rs} \Delta \eta^s + A^r P^{rs} \eta^s + (\eta^+)^r P^{rs} (A^+)^s] \equiv \\ \equiv E\{\exp[\tilde{A}^s \hat{\eta}^s + (\hat{\eta}^+)^s (\tilde{A}^+)^s]\} = \exp[-i\tilde{A}^r \Delta^{-1} (\tilde{A}^+)^s] \end{aligned} \quad (18)$$

where we denoted

$$\tilde{A}^r \equiv P^{rs} A^s, \quad (\tilde{A}^+)^r \equiv P^{rs} (A^+)^s \quad (19)$$

and the normalization factor N is such that

$$\begin{aligned} E(1) &= 1 \\ E[\hat{\eta}^r(\mathbf{x})] &= E[(\hat{\eta}^+)^r(\mathbf{x})] = 0 \\ E[\hat{\eta}^r(\mathbf{x})(\hat{\eta}^+)^s(\mathbf{y})] &= \frac{\delta^2}{\delta(\tilde{A}^+)^r(\mathbf{x}) \delta \tilde{A}^s(\mathbf{y})} \exp[-i(\tilde{A}^+)^r \Delta^{-1} \tilde{A}^s]_{|\tilde{A}=\tilde{A}^+=0} = \\ &= \delta^{rs} i(4\pi|\mathbf{x}-\mathbf{y}|)^{-1} \end{aligned} \quad (20)$$

Upon the substitution

$$A^r = \sqrt{\varepsilon} P^{rs} \frac{\delta}{\delta g^s}, \quad (A^+)^r = \sqrt{\varepsilon} P^{rs} \frac{\delta}{\delta (g^+)^s} \quad (21)$$

we obtain the needed representation in the quantum (exotic) stochastic terms

$$\exp[-i\varepsilon \frac{\delta}{\delta (g^+)^r} \Delta^{-1} P^{rs} \frac{\delta}{\delta g^s}] = E\{\exp[\sqrt{\varepsilon} P^{rs} \hat{\eta}^s \frac{\delta}{\delta (g^+)^r} + \sqrt{\varepsilon} P^{rs} (\hat{\eta}^+)^s \frac{\delta}{\delta (g^+)^r}]\} \quad (22)$$

We insert this formula into (15), and repeat the steps from the previous sections including the use of an identity

$$\exp(\mathbf{u} \cdot \frac{\delta}{\delta \mathbf{g}} + \mathbf{u}^+ \cdot \frac{\delta}{\delta \mathbf{g}^+}) \exp[F(\mathbf{g}, \mathbf{g}^+)] = \exp[F(\mathbf{g} + \mathbf{u}, \mathbf{g}^+ + \mathbf{u}^+)] \exp(\mathbf{u} \cdot \frac{\delta}{\delta \mathbf{g}} + \mathbf{u}^+ \cdot \frac{\delta}{\delta \mathbf{g}^+}) \quad (23)$$

In our case at hand

$$u_l^r = \sqrt{\varepsilon} P^{rs} \hat{\eta}_l^s, \quad (u_l^+)^r = \sqrt{\varepsilon} P^{rs} (\hat{\eta}_l^+)^s, \quad k, l = 1, 2, \dots, n \quad r, s = 1, 2, 3$$

$$F = -i\tilde{V} \quad (24)$$

where \tilde{V} is given in (16).

The evolution operator takes on the form

$$U_t =$$

$$= \lim_{n \rightarrow \infty} E \{ \exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\mathbf{g} + \hat{\mathbf{C}}_k - \hat{\mathbf{C}}_n, \mathbf{g}^+ + \hat{\mathbf{C}}_k^+ - \hat{\mathbf{C}}_n^+)] \exp(-\hat{\mathbf{C}}_n \cdot \frac{\delta}{\delta \mathbf{g}} - \hat{\mathbf{C}}_n^+ \cdot \frac{\delta}{\delta \mathbf{g}^+}) \} \quad (25)$$

where the notations are

$$\hat{\mathbf{C}}_k(\mathbf{x}) \equiv -\sqrt{\varepsilon} \sum_{l=1}^k P \hat{\eta}_l(\mathbf{x}), \quad \hat{\mathbf{C}}_k^+(\mathbf{x}) \equiv -\sqrt{\varepsilon} \sum_{l=1}^k P \hat{\eta}_l^+(\mathbf{x}) \quad (26)$$

It is convenient to work with the interpolating quantum (exotic) random fields

$$\hat{\mathbf{C}}(\mathbf{x}, k\varepsilon) = \hat{\mathbf{C}}_k(\mathbf{x}), \quad \hat{\mathbf{C}}^+(\mathbf{x}, k\varepsilon) = \hat{\mathbf{C}}_k^+(\mathbf{x}), \quad k = 1, 2, \dots, n \quad (27)$$

Using (20) we find

$$E[\hat{\mathbf{C}}(\mathbf{x}, s)] = E[\hat{\mathbf{C}}^+(\mathbf{x}, s)] = 0, \quad 0 \leq s \leq t$$

$$E[\hat{\mathbf{C}}^r(\mathbf{x}, j\varepsilon)(\hat{\mathbf{C}}^+)^s(\mathbf{y}, k\varepsilon)] = j \wedge k \varepsilon \frac{i}{4\pi} |\mathbf{x} - \mathbf{y}|^{-1} \quad (28)$$

Acting with the evolution operator U_t on some initial state $\psi_0(\mathbf{g}, \mathbf{g}^+)$ we get

$$\psi_t(\mathbf{g}, \mathbf{g}^+) = U_t \psi_0(\mathbf{g}, \mathbf{g}^+) \equiv$$

$$\equiv \int \prod_{x,r} dg^{lr}(\mathbf{x}) d(g^{+l})^r(\mathbf{x}) \tilde{K}(\mathbf{g}, \mathbf{g}^+, t; \mathbf{g}', \mathbf{g}^{+'}, 0) \psi_0(\mathbf{g}', \mathbf{g}^{+'}) \quad (29)$$

where the propagator, for $t \geq 0$, is given by the equivalent formulae

$$\tilde{K}(\mathbf{g}, \mathbf{g}^+, t; \mathbf{g}', \mathbf{g}^{+'}, 0) =$$

$$= \lim_{n \rightarrow \infty} E \{ \exp[-i\varepsilon \sum_{k=1}^n \tilde{V}(\mathbf{g}' + \hat{\mathbf{C}}_k, \mathbf{g}^{+'} + \hat{\mathbf{C}}_k^+)] \delta(\mathbf{g} - \mathbf{g}' - \hat{\mathbf{C}}_n) \delta(\mathbf{g}^+ - \mathbf{g}^{+'} - \hat{\mathbf{C}}_n^+) \} =$$

$$= E \{ \exp[-i \int_0^t ds \tilde{V}(\mathbf{g}' + \hat{\mathbf{C}}_s, \mathbf{g}^{+'} + \hat{\mathbf{C}}_s^+)] \delta(\mathbf{g} - \mathbf{g}' - \hat{\mathbf{C}}_t) \delta(\mathbf{g}^+ - \mathbf{g}^{+'} - \hat{\mathbf{C}}_t^+) \} = \quad (30)$$

We perform a change of the variables of integration as follows

$$\mathbf{g}_k = \mathbf{g}' - \sqrt{\varepsilon P}(\boldsymbol{\eta}_1 + \dots + \boldsymbol{\eta}_k), \quad \mathbf{g}_k^+ = \mathbf{g}^{+'} - \sqrt{\varepsilon} P(\boldsymbol{\eta}_1^+ + \dots + \boldsymbol{\eta}_k^+) \quad k = 0, 1, \dots, n \quad (31)$$

$$\mathbf{g}_0 = \mathbf{g}', \quad \mathbf{g}_0^+ = \mathbf{g}^{+'} \quad (\text{both transversal, cf. (7)})$$

We also introduce two interpolating functions $\mathbf{g}(\mathbf{x}, s)$, $\mathbf{g}^+(\mathbf{x}, s)$ coinciding with \mathbf{g}_k , \mathbf{g}_k^+ , correspondingly when $s = k\varepsilon$. In the continuum limit when $\varepsilon \rightarrow 0$, we get

$$\mathbf{g}(\mathbf{x}, s) = \mathbf{g}'(\mathbf{x}) + \mathbf{C}(\mathbf{x}, s), \quad \mathbf{g}^+(\mathbf{x}, s) = \mathbf{g}^{+'}(\mathbf{x}) + \mathbf{C}^+(\mathbf{x}, s) \quad (32)$$

Both fields are transversal by construction, cf. (26). Inverting the equations (32), we get

$$-P\boldsymbol{\eta}_k = \frac{1}{\sqrt{\varepsilon}}(\mathbf{g}_k - \mathbf{g}_{k-1}), \quad P\boldsymbol{\eta}_k^+ = \frac{1}{\sqrt{\varepsilon}}(\mathbf{g}_k^+ - \mathbf{g}_{k-1}^+), \quad k = 0, 1, \dots, n \quad (33)$$

Absolute values of the Jacobians are

$$\left| \frac{\partial P(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_n)}{\partial(\mathbf{g}_1, \dots, \mathbf{g}_n)} \right| = \left| \frac{\partial P\boldsymbol{\eta}_1^+, \dots, P\boldsymbol{\eta}_n^+}{\partial \mathbf{g}_1^+, \dots, \mathbf{g}_n^+} \right| = \varepsilon^{-n/2} \quad (34)$$

In terms of the new variables, the formula for \tilde{K} becomes

$$\begin{aligned} \tilde{K}(\mathbf{g}, \mathbf{g}^+, t; \mathbf{g}', \mathbf{g}^{+'}, 0) &= \lim_{n \rightarrow \infty} \prod_{k=1}^n \left[\frac{1}{N} \int \prod_{\mathbf{x}, s} \frac{d\mathbf{g}_k^r(\mathbf{x}, s)}{\sqrt{\varepsilon}} \frac{d(\mathbf{g}_k^+(\mathbf{x}, s))^r}{\sqrt{\varepsilon}} \right] \times \\ &\times \exp\{-i\varepsilon \sum_{k=1}^n \left[\frac{1}{\varepsilon^2}(\mathbf{g}_k^+ - \mathbf{g}_{k-1}^+) \cdot \Delta(\mathbf{g}_k - \mathbf{g}_{k-1}) - \tilde{V}(\mathbf{g}, \mathbf{g}^+) \right]\} \delta(\mathbf{g} - \mathbf{g}_n) \delta(\mathbf{g}^+ - \mathbf{g}_n^+) = \end{aligned} \quad (35)$$

$$\begin{aligned} &= \frac{1}{N} \int \prod_{\mathbf{x}, s, r} \frac{d\mathbf{g}^r(\mathbf{x}, s)}{0} \frac{d(\mathbf{g}^+)^r(\mathbf{x}, s)}{0} \times \\ &\times \exp\{-i \int d\mathbf{x} \int_0^t ds \partial_\mu \mathbf{g}^+(\mathbf{x}, s) \cdot \Delta \partial^\mu \mathbf{g}(\mathbf{x}, s)\} \delta(\mathbf{g} - \mathbf{g}_t) \delta(\mathbf{g}^+ - \mathbf{g}_t^+) |_{\mathbf{g}_0 = \mathbf{g}', \mathbf{g}_0^+ = \mathbf{g}^{+'}} = \\ &\equiv K(\nabla \times \mathbf{g}, \nabla \times \mathbf{g}^+, t; \nabla \times \mathbf{g}', \nabla \times \mathbf{g}^{+'}, 0) \end{aligned} \quad (36)$$

Notice that in the exponent we got the gauge-invariant Lagrangian, cf. (9). Going though the time variable change as in (A36), we conclude that

$$\tilde{K}(\mathbf{g}, \mathbf{g}^+, t; \mathbf{g}', \mathbf{g}^{+'}, 0) = \tilde{K}(\mathbf{g}, \mathbf{g}^+, t/2; \mathbf{g}', \mathbf{g}^{+'}, -t/2) = \quad (37)$$

$$= E\{\exp[-i \int_{-t/2}^{t/2} ds \tilde{V}(\mathbf{g}' + \hat{\mathbf{C}}_{s+t/2}, \mathbf{g}^{+'} + \hat{\mathbf{C}}_{s+t/2}^+)] \delta(\mathbf{g} - \mathbf{g}' - \hat{\mathbf{C}}_t) \delta(\mathbf{g}^+ - \mathbf{g}^{+'} - \hat{\mathbf{C}}_t^+)\} \quad (38)$$

Replacing \tilde{V} with \tilde{V}_{J,J^+} in the last formula where

$$\tilde{V}_{J,J^+} \equiv \tilde{V} - \mathbf{J}_{s+l/2} \cdot \hat{\mathbf{C}}_{s+l/2} - \mathbf{J}_{s+l/2}^+ \cdot \hat{\mathbf{C}}_{s+l/2}^+ \quad (39)$$

we obtain the propagator \tilde{K}_{J,J^+} .

THE $Z(J, J^+)$ GENERATING FUNCTIONAL

Considering $\tilde{K}(\mathbf{0}, \mathbf{0}, \infty; \mathbf{0}, \mathbf{0}, -\infty)$, we encounter an integral over a gauge-invariant measure from a gauge-invariant action in the exponent. The Faddeev-Popov ansatz, [12], of extracting out a volume of the gauge group is utilized. This leads to the result

$$\tilde{Z}(J, J^+) \equiv Z(J, J^+) = \quad (40)$$

$$= \frac{1}{N} \int \prod_{\mathbf{x}, s} d\mathbf{g}(\mathbf{x}, s) d\mathbf{g}^+(\mathbf{x}, s) \delta[\nabla \cdot \mathbf{g}(\mathbf{x}, s)] \delta[\nabla \cdot \mathbf{g}^+(\mathbf{x}, s)] \exp\{i \int dx (L + \mathbf{J} \cdot \mathbf{g} + \mathbf{J}^+ \cdot \mathbf{g}^+)\}$$

The normalization constant N has been modified accordingly so that $Z(\mathbf{0}, \mathbf{0}) = 1$.

Shifting the variables and eliminating the linear terms in the fields, we find for $Z(J, J^+)$

$$Z(J, J^+) = \exp(i \int d^4x \int d^4y J^{\mu\nu}(x) P^{\mu\nu} D_F(x-y; 0) J^s(y)) \quad (41)$$

where we denoted

$$D_F(x-y; 0) = -(\frac{1}{2\pi})^4 \int d^4k (k^2 + i\varepsilon)^{-1} \exp[ik(x-y)] \quad (42)$$

$$= i(\frac{1}{2\pi})^3 \int \frac{d\mathbf{k}}{2\omega_k} \exp[-i\omega_k|x^0 - y^0| + i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})] \quad (43)$$

$$= \{4\pi^2 i [(x-y)^2 - i\varepsilon]\}^{-1} \quad (44)$$

$$\omega_k \equiv |\mathbf{k}|$$

The choice of the Feynman function corresponds to the **radiation conditions**:

Only incoming wave at $x^0 - y^0 \rightarrow -\infty$, and only outgoing wave at $x^0 - y^0 \rightarrow \infty$.

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Both Parts are published online. Website: Exotic Probability Theories and Quantum Mechanics References

http://physics.bu.edu/~youssef/quantum/quantum_refs.html

A parallel between quantum theory and probability theory has captured my attention for quite some time. I think that a former might gain greatly from fast developping probabilistic methods. It suffice to say that a turning away from them means turning a blind eye to about **one third of volume of the modern mathematics**.

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