Photon's Propagator From Maxwell Electrodynamics And The Quantum Cauchy Stochastic Process

W. Garczynski*
Retired professor of the Institute of Theoretical Physics, University of Wroclaw, Pl. Maxa Borna 9, 50-204 Wroclaw, Poland

Starting from Maxwell equations the single photon propagator has been found as the transition amplitude density for, so called, quantum Cauchy stochastic process. The bifurcation point between the classical and the quantum descriptions of light is located.

INTRODUCTION

Maxwell electrodynamics [1], [2], [3] is not only relativistic but also quantum as far as the pure electromagnetic field [6], [8-15]. We shall review briefly the main arguments leading to this conclusion, and go one step further by calculating the transition amplitude for the photon. We leave aside the multi-photon systems and the issue of accuracy of the classical description of strong electromagnetic fields.

In the Gaussian system of units the "microscopic" Maxwell equations for the pure photon field are

$$\nabla \times \mathbf{E} + \tfrac{1}{c}\partial_t \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{B} - \tfrac{1}{c}\partial_t \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (1.1)$$

We want to stress that they are the basic fields of a single photon, and not some averages over regions of space or time intervals.

The equations can be compactly written in terms of, so called, Riemann-Silberstein [4-5] complex vector

$$\mathbf{f} = \mathbf{E} + i\mathbf{B} \qquad (1.2)$$

namely

$$\partial_t \mathbf{f} = -ic\nabla \times \mathbf{f}$$

$$\nabla \cdot \mathbf{f} = 0 \qquad (1.3)$$

*Present address: 47 Country View Lane, Middle Island, NY 11953, USA
E-mail: jgarchin@optonline.net

Furthermore, we use the identity for the curl of a vector [17]

$$\nabla \times \mathbf{f} = -i(\mathbf{S} \cdot \nabla)\mathbf{f} \tag{1.4}$$

where S^k are the spin-1 matrices

$$(S^k)_{lm} = -i\varepsilon_{klm} \qquad k,l,m = 1,2,3 \tag{1.5}$$

and where ε_{klm} are the fully antisymmetric Levi-Civita tensor with $\varepsilon_{123} = 1$,

$$S^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad S^3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.6}$$

Multiplying both sides by $i\hbar$, we get the Schroedinger equation [6]

$$i\hbar\partial_t \mathbf{f} = c(\mathbf{S} \cdot \mathbf{p})\mathbf{f} \qquad \mathbf{p} \equiv -i\hbar\nabla \tag{1.7}$$

$$\nabla \cdot \mathbf{f} = 0$$

Notice **the Planck constant** $\hbar$ **cancels on both sides** and, in fact, might be eliminated from the calculations as it is in Maxwell electrodynamics. It appears to be **the reason** for the known **classic-quantum duality** in description of radiation. Recall, in this regard, a lamentation by Albert Einstein uttered in 1924:

"There are therefore now two theories of light, both indispensable and as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists-without any logical connections"

The Schroedinger equation identifies the Hamiltonian [16] and thus opens the door to the first-quantized theory of photon

$$\mathrm{H} = c(\mathbf{S} \cdot \tfrac{\hbar}{i}\nabla) = c\hbar \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \tag{1.8}$$

We denote the helicity matrix of photon

$$\Lambda = (\mathbf{S} \cdot \tfrac{\hbar}{i}\nabla)(\hbar\sqrt{-\triangle})^{-1} \tag{1.9}$$

it commutes with the Hamiltonian so the helicity is conserved, and both matrices share their eigenvectors. Since the total energy is conserved, we have

$$\int d^3 r \mathbf{f}^{+}\mathbf{f} = \int d^3 r(\mathbf{E}^2 + \mathbf{B}^2) = 8\pi E \tag{1.10}$$

The last formula can be rewritten in a conventional manner if the RS vector is replaced with LP one proposed by Landau and Peierls [8]. In the momentum space and with $(p \equiv |\mathbf{p}|)$

$$\mathbf{f}_{RS} = (8\pi c p)^{1/2}\mathbf{f}_{LP} \tag{1.11}$$

The total energy is now given by

$$\int c p \mathbf{f}^{+}_{LP}\mathbf{f}_{LP} d^3 p = E \tag{1.12}$$

but the connection between the LP and the RS vectors, in the coordinate space, becomes non-local [10]. It is immaterial for our purposes which vector to use. Point is that they both identify the same Hamiltonian which is the departure point for our considerations. We shall stick with the RS vector as the simpler one.

Following [12], we consider the 6-dimensional vector F as representing the photon system

$$F = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} \mathbf{f} \\ \mathbf{f}^{*} \end{array}\right\} \tag{1.13}$$

It satisfies the equations

$$i\hbar\partial_t F = HF$$

$$\nabla \cdot F = 0 \tag{1.14}$$

where the 6×6 resulting Hamiltonian is of the block-diagonal form

$$H = \begin{bmatrix} \mathrm{H} & \mathbf{0} \\ \mathbf{0} & -\mathrm{H} \end{bmatrix} \tag{1.15}$$

and the 6-dimensional divergence is understood as follows

$$\nabla \cdot F = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} \nabla \cdot \mathbf{f} \\ -\nabla \cdot \mathbf{f}^{*} \end{array}\right\} = \mathbf{0} \tag{1.16}$$

With the conventions

$$ct \equiv x^0, \qquad x \equiv x^1, \qquad y \equiv x^2, \qquad z \equiv x^3$$

$$\tfrac{E}{c} \equiv p^0, \qquad p_x \equiv p^1, \qquad p_y \equiv p^2, \qquad p_z \equiv p^3 \tag{1.17}$$

$$E = cp, \qquad (g^{\mu\nu}) = diag(1,-1,-1,-1)$$

we introduce the electromagnetic tensor

$$(f^{\mu\nu}) = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix} = (\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1.18}$$

The RS vector can be expressed in various equivalent ways

$$F = \frac{1}{\sqrt{2}} \left\{ \begin{matrix} \mathbf{E} + i\mathbf{B} \\ \mathbf{E} - i\mathbf{B} \end{matrix} \right\} = \frac{1}{\sqrt{2}} \begin{bmatrix} f^{10} + if^{32} \\ f^{20} + if^{13} \\ f^{30} + if^{21} \\ f^{10} - if^{32} \\ f^{20} - if^{13} \\ f^{30} - if^{21} \end{bmatrix} \tag{1.19}$$

Expressing tensor $f^{\mu\nu}$ through the 4-vector potentials A^μ we get yet another formula for F.

2. EIGENVALUES AND EIGENVECTORS OF THE HAMILTONIAN

The Hamiltonian (1.8) does not depend on time explicitly, hence we may present a solution of the Schroedinger equation (1.14) as the product (variables separation)

$$F(\mathbf{r}, \mathbf{t}) = G(\mathbf{r}) \exp\{-\tfrac{i}{h} Et\}, \qquad G = \frac{1}{\sqrt{2}} \left\{ \begin{matrix} \mathbf{g} \\ \mathbf{g}^* \end{matrix} \right\} \tag{2.1}$$

where G solves the eigenvalue equation

$$HG = EG \tag{2.2}$$

which splits into two three-dimensional eigenvalue problems

$$\mathsf{H}\mathbf{g} = E\mathbf{g}, \qquad \mathsf{H}\mathbf{g}^* = -E\mathbf{g}^* \tag{2.3}$$

Since H contains spatial derivatives, it is convenient to analyze the equations in the momentum space

$$\mathbf{g}(\mathbf{r}) = \int d^3p(2\pi\hbar)^{-3/2}\mathbf{g}(\mathbf{p})\exp\{\tfrac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}\} \tag{2.4}$$

Applying the operator H as given in (1.8), we obtain

$$ic\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}\mathbf{g} = E\mathbf{g} \tag{2.5}$$

Hence the eigenvector $\mathbf{g}$ satisfies the homogeneous algebraic equations

$$\mathsf{A}\mathbf{g} = \mathbf{0} \tag{2.6}$$

with the matrix

$$\mathsf{A} \equiv \begin{bmatrix} -\frac{E}{c} & -ip_z & ip_y \\ ip_z & -\frac{E}{c} & -ip_x \\ -ip_y & ip_x & -\frac{E}{c} \end{bmatrix} \tag{2.7}$$

Its determinant must vanish for non-zero solution of (2.6) to exist

$$\det\mathsf{A} = \tfrac{E}{c}(p - \tfrac{E}{c})(p + \tfrac{E}{c}) = 0 \tag{2.8}$$

The eigenvalues are obvious

$$\begin{aligned} &1. \quad E = 0 \\ &2. \quad E = cp \\ &3. \quad E = -cp \end{aligned} \tag{2.9}$$

Consider the zero eigenvalue first. Its corresponding eigenvector $\mathbf{g}_0$ satisfies the equation

$$\begin{bmatrix} 0 & -ip_z & ip_y \\ ip_z & 0 & -ip_x \\ -ip_y & ipx & 0 \end{bmatrix} \mathbf{g}_0 = \mathbf{0} \tag{2.10}$$

Solution is of the form

$$\mathbf{g}_0(\mathbf{p}) = \alpha(p_z)^{-1}\mathbf{p} \tag{2.11}$$

where α is an arbitrary constant. It is ruled out by the transversality condition

$$\mathbf{p} \cdot \mathbf{g}_0(\mathbf{p}) \equiv 0 \qquad \text{for any } \mathbf{p} \neq \mathbf{0}$$

which leads to $\alpha = 0$.

Consider now the eigenvalue $E = cp$ and the corresponding equation

$$\begin{bmatrix} -p & -ip_z & ip_y \\ ip_z & -p & -ip_x \\ -ip_y & ip_x & -p \end{bmatrix} \mathbf{g} = \mathbf{0} \tag{2.12}$$

The normalized solution is given by

$$\mathbf{g} \equiv \mathbf{e}(\mathbf{p}) = (pp_\perp\sqrt{2})^{-1} \begin{bmatrix} -p_x p_z + ipp_y \\ -p_y p_z - ipp_x \\ p_\perp^2 \end{bmatrix} \tag{2.13}$$

$$p_\perp = (p_x^2 + p_y^2)^{1/2}$$

$$\mathbf{e} \cdot \mathbf{e} \equiv \mathbf{e}^+\mathbf{e} = 1 \tag{2.14}$$

Also, the vector $\mathbf{e}$ came out transversal, automatically

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}) = 0 \tag{2.15}$$

Thus we have constructed a solution of the Schroedinger equation

$$\mathbf{f}_p(\mathbf{r}, t) = f(\mathbf{p})\mathbf{e}(\mathbf{p}) \exp\{-\tfrac{i}{h}(cpt - \mathbf{p} \cdot \mathbf{r})\} \tag{2.16}$$

for any value of $\mathbf{p}$. The vectors $\mathbf{e}, \mathbf{e}^*$ are called the **circular polarization vectors**,

($\mathbf{e} \backsim$ Left, $\mathbf{e}^* \sim$ Right polarization for a photon, and reverse, for an anti-photon).

For the third eigenvalue $E = -cp$ we get the eigenvector equation

$$\begin{bmatrix} p & -ip_z & ip_y \\ ip_z & p & -ip_x \\ -ip_y & ip_x & p \end{bmatrix} \mathbf{g}' = \mathbf{0} \tag{2.17}$$

Its normalized solution is

$$\mathbf{g}' \equiv \mathbf{e}^*(\mathbf{p}) = (pp_\perp \sqrt{2}\,)^{-1} \begin{bmatrix} -p_x p_z - ipp_y \\ -p_y p_z + ipp_x \\ p_\perp^2 \end{bmatrix} = \mathbf{e}(-\mathbf{p}) \tag{2.18}$$

We get another solution to the Schroedinger equation for any $\mathbf{p}$

$$\mathbf{f}_p'(\mathbf{r},t) = f^*(\mathbf{p})\mathbf{e}^*(\mathbf{p})\exp\{\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \;\; = \mathbf{f}_p^*(\mathbf{r},t) \tag{2.19}$$

We can obtain a general solution by taking linear combinations of the particular solutions (2.16), (2.19)

$$\mathbf{f}(\mathbf{r},t) = \int d^3p(2\pi\hbar)^{-3/2}\mathbf{e}(\mathbf{p})[f(\mathbf{p})\exp\{-\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \; + f^*(-\mathbf{p})\exp\{\tfrac{i}{\hbar}(cpt + \mathbf{p}\cdot\mathbf{r})\}] \tag{2.20}$$

It is obvious that the wave-vector also satisfies the d'Alembert equation

$$\Box\mathbf{f} \equiv (\partial_0^2 - \triangle)\mathbf{f} \equiv \mathbf{0} \tag{2.21}$$

In fact any solution of the Schroedinger equation (1.7) **automatically** satisfies the d'Alembert equation as it can be seen by taking the second time derivative and using the identity curlcurl = graddiv - $\triangle$, or by manipulating the spin matrices [12].

The helicity matrix(projection of the spin on the momentum) is

$$\Lambda \equiv (\mathbf{S}\cdot\mathbf{p})p^{-1} = \mathrm{H}(cp)^{-1} = \frac{1}{\sqrt{-\triangle}} \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \equiv \Lambda(\partial) \tag{2.22}$$

Since Λ commutes with H, it is conserved and both matrices share the same eigenvectors. Acting on the vector $\mathbf{f}$ we find that

$$\Lambda\mathbf{f}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \tag{2.23}$$

It follows from the following observations

$$\Lambda(\partial)\exp\{\tfrac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}\} = \exp\{\tfrac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}\}\Lambda_p \tag{2.24}$$

where

$$\Lambda_p \equiv p^{-1}\begin{bmatrix} 0 & -ip_z & ip_y \\ ip_z & 0 & -ip_x \\ -ip_y & ip_x & 0 \end{bmatrix} \tag{2.25}$$

and, according to (2.12)

$$\Lambda_p\mathbf{e}(\mathbf{p}) = \mathbf{e}(\mathbf{p}) \tag{2.26}$$

As the result, we conclude that the vector $\mathbf{f}$ has helicity plus one

$$\Lambda\mathbf{f}(\mathbf{r},t) = \mathbf{f}(\mathbf{r},t) \tag{2.27}$$

Similarly, the complex conjugated vector $\mathbf{f}^*$ also has helicity plus one

$$\Lambda_p\mathbf{e}^*(\mathbf{p}) = -\mathbf{e}^*(\mathbf{p}) \tag{2.28}$$

and, as the result

$$\Lambda\mathbf{f}^*(\mathbf{r},t) = \mathbf{f}^*(\mathbf{r},t) \tag{2.29}$$

We would like to presentt the main relations concerning the polarization vectors

$$\begin{gathered}
\mathbf{e}\cdot\mathbf{e} = \mathbf{e}^*\cdot\mathbf{e}^* = 1 \\
\mathbf{e}\cdot\mathbf{e}^* = \mathbf{e}^*\cdot\mathbf{e} = 0 \\
\mathbf{e}\times\mathbf{e}^* = i(p)^{-1}\mathbf{p} \\
\mathbf{e}\cdot\mathbf{p} = \mathbf{e}^*\cdot\mathbf{p} = 0 \qquad (2.30) \\
\mathbf{p}\times\mathbf{e} = -ip\mathbf{e}, \qquad \mathbf{p}\times\mathbf{e}^* = ip\mathbf{e}^* \\
\mathbf{e} = \tfrac{1}{\sqrt{2}}(\mathbf{l}_1 + i\mathbf{l}_2), \qquad \mathbf{e}^* = \tfrac{1}{\sqrt{2}}(\mathbf{l}_1 - i\mathbf{l}_2)
\end{gathered}$$

where $\mathbf{l}_1, \mathbf{l}_2$ are the **linear polarization vectors**

$$\mathbf{l}_1 \equiv (pp_\perp)^{-1}\begin{bmatrix} -p_x p_z \\ -p_y p_z \\ p_\perp^2 \end{bmatrix}, \qquad \mathbf{l}_2 \equiv (pp_\perp)^{-1}\begin{bmatrix} pp_y \\ -pp_x \\ 0 \end{bmatrix} \tag{2.31}$$

The following relations hold

$$\begin{aligned} &\mathbf{l}_1 \cdot \mathbf{l}_1 = \mathbf{l}_2 \cdot \mathbf{l}_2 = 1 \\ &\mathbf{l}_1 \cdot \mathbf{l}_2 = 0 \\ &\mathbf{l}_1 \times \mathbf{l}_2 = (p)^{-1}\mathbf{p} \\ &\mathbf{p} \cdot \mathbf{l}_1 = \mathbf{p} \cdot \mathbf{l}_2 = 0 \\ &\mathbf{p} \times \mathbf{l}_1 = p\mathbf{l}_2 \\ &\mathbf{p} \times \mathbf{l}_2 = -p\mathbf{l}_1 \\ &\mathbf{l}_1(-\mathbf{p}) = \mathbf{l}_1(\mathbf{p}) \\ &\mathbf{l}_2(-\mathbf{p}) = -\mathbf{l}_2(\mathbf{p}) \end{aligned} \tag{2.32}$$

We now define the wave-vectors for the photon and the anti-photon with definite helicities. For this we split the formula (2.20) for $\mathbf{f}(\mathbf{r},t)$ into positive and negative frequency parts. For instance, the vector

$$\langle \mathbf{r}|\boldsymbol{\gamma}_1(t)\rangle \equiv \int d^3p(2\pi\hbar)^{-3/2} f(\mathbf{p})\mathbf{e}(\mathbf{p})\exp\{-\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \tag{2.33}$$

represents a photon of helicity plus one. Similarly the vector

$$\langle \mathbf{r}|\bar{\boldsymbol{\gamma}}_1(t)\rangle \equiv \int d^3p(2\pi\hbar)^{-3/2} f^*(\mathbf{p})\mathbf{e}^*(\mathbf{p})\exp\{\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \tag{2.34}$$

represents an anti-photon with the same helicity plus one. The vector taken from $\mathbf{f}(-\mathbf{r},-t)$ represents a photon of helicity minus one

$$\langle \mathbf{r}|\boldsymbol{\gamma}_{-1}(t)\rangle \equiv \int d^3p(2\pi\hbar)^{-3/2} f(\mathbf{p})\mathbf{e}(\mathbf{p})\exp\{\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \tag{2.35}$$

Finally, we get the vector representing an anti- photon with helicity minus one

$$\langle \mathbf{r}|\bar{\boldsymbol{\gamma}}_{-1}(t)\rangle \equiv \int d^3p(2\pi\hbar)^{-3/2} f^*(\mathbf{p})\mathbf{e}^*(\mathbf{p})\exp\{-\tfrac{i}{\hbar}(cpt - \mathbf{p}\cdot\mathbf{r})\} \tag{2.36}$$

We impose the following ortho-normalization conditions using (2.30)

$$\langle \boldsymbol{\gamma}_1|\boldsymbol{\gamma}_1\rangle = \langle \boldsymbol{\gamma}_{-1}|\boldsymbol{\gamma}_{-1}\rangle = \langle \bar{\boldsymbol{\gamma}}_1|\bar{\boldsymbol{\gamma}}_1\rangle = \langle \bar{\boldsymbol{\gamma}}_{-1}|\bar{\boldsymbol{\gamma}}_{-1}\rangle = \int d^3p|f(\mathbf{p})|^2 = 8\pi E$$

$$\langle \boldsymbol{\gamma}_1|\boldsymbol{\gamma}_{-1}\rangle = \langle \bar{\boldsymbol{\gamma}}_1|\bar{\boldsymbol{\gamma}}_{-1}\rangle = 0 \tag{2.37}$$

Having the 3-dimensional vectors we may construct the full 6-dimensional transversal vectors representing the photon system of definite helicity

$$F_\lambda = \frac{1}{\sqrt{2}}\begin{Bmatrix} \boldsymbol{\gamma}_\lambda \\ \bar{\boldsymbol{\gamma}}_\lambda \end{Bmatrix} \equiv |\lambda\rangle$$

where (2.38)

$$\lambda = \pm 1$$

They do satisfy the Schroedinger equations

$$i\hbar\partial_t F_\lambda = HF_\lambda \tag{2.39}$$

and the normalization conditions

$$\langle\lambda|\lambda'\rangle = (8\pi E)\delta_{\lambda\lambda'} \qquad \lambda,\lambda' = \pm 1$$

(2.40)

$$\langle\lambda|-\lambda\rangle = 0$$

Notice, please, that all the polarization vectors are scale-invariant

$$\mathbf{e}(\mathbf{p}) = \mathbf{e}(\hbar\mathbf{k}) = \mathbf{e}(\mathbf{k}) \qquad (\text{ since } \hbar \text{ is positive }) \tag{2.41}$$

and the same holds for the complex conjugateded vector $\mathbf{e}^*$, and for their real and imaginary parts $\mathbf{l}_j(\mathbf{p})$ $\quad j = 1,2$. Hence, the same polarization vector enters a solution of the d'Alembert equation using the wave-vectors $\mathbf{k}$ (no $\hbar$) (cf. e.g., [15], formula (56)) and the solution (2.20) of the Schroedinger equation that uses the momenta $\mathbf{p}$, and the Planck constant $\hbar$ appears

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3. THE EVOLUTION MATRIX

According to the Schroedinger equation (1.14), we have

$$F(t) = U(t)F(0)$$

where

$$U(t) = \exp\{-\tfrac{i}{\hbar}Ht\} \tag{3.1}$$

with H in the block-diagonal form as shown in (1.15). We find for the powers of H

$$H^{2n} = \begin{bmatrix} \mathsf{H}^{2n} & \mathbf{0} \\ \mathbf{0} & \mathsf{H}^{2n} \end{bmatrix}, \qquad H^{2n+1} = \begin{bmatrix} \mathsf{H}^{2n+1} & \mathbf{0} \\ \mathbf{0} & -\mathsf{H}^{2n+1} \end{bmatrix} \tag{3.2}$$

$$n = 0, 1, 2, \ldots$$

The evolution matrix also has the block-diagonal form and simplifies greatly when applied to a state of definite helicity(see APPENDIX),

$$U(t) = \begin{bmatrix} \mathrm{U}(t) & \mathbf{0} \\ \mathbf{0} & \mathrm{U}^{+}(t) \end{bmatrix} \tag{3.3}$$

with

$$\mathrm{U}(t) = \exp\{-\tfrac{i}{\hbar}\mathsf{H}t\}$$

Consider, for example, two states of helicity λ which may differ by the different functions $f(\mathbf{p})$, and evaluate the matrix element

$$\langle\lambda|U(t)|\lambda\rangle' = \sum_{j,k=1}^{6} {}_j\langle\lambda|U_{jk}(t)|\lambda\rangle_k' =$$

$$= \sum_{j,k=1}^{6} \int d^3r d^3r' \langle\lambda|\mathbf{r}\rangle_j \langle\mathbf{r}|U_{jk}(t)|\mathbf{r}'\rangle\langle\mathbf{r}'|\lambda\rangle_k' \tag{3.4}$$

We replace the helicity operator Λ by its eigenvalue $\lambda\mathbf{1}_3$ inside U(t). Next, we calculate the inside matrix element in the momentum space, and again replace the momentum operator with the relevant eigenvalue

$$\langle\mathbf{r}|U_{jk}(t)|\mathbf{r}'\rangle = \int d^3p d^3p' (2\pi\hbar)^{-3} \exp\{\tfrac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - \mathbf{p}'\cdot\mathbf{r}')\}\langle\mathbf{p}|U_{jk}(t)|\mathbf{p}'\rangle \tag{3.5}$$

where the Hamiltonian is replaced by the simpler matrix

$$\mathrm{H} = c\hbar\sqrt{-\triangle}\,\Lambda \sim cp'\lambda\mathbf{1}_3 \tag{3.6}$$

The variable λ can be associated with the time variable t since the Hamiltonian is multiplied by it. With this in mind, we write

$$\langle \mathbf{p}|U_{jk}(t)|\mathbf{p}'\rangle = \begin{bmatrix} \exp\{-\frac{i}{h}\lambda tcp\}\mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & \exp\{\frac{i}{h}\lambda tcp\}\mathbf{1}_3 \end{bmatrix}_{jk} \delta(\mathbf{p}-\mathbf{p}') \qquad (3.7)$$

Hence, we may perform the integration over the momenta in (3.5) passing to the spherical coordinates and regularizing the integrals over p

$$\langle \mathbf{r}|U_{jk}(t)|\mathbf{r}'\rangle = \begin{bmatrix} C(\mathbf{r},\lambda t;\mathbf{r}',0)\mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & C^*(\mathbf{r},\lambda t;\mathbf{r}',0)\mathbf{1}_3 \end{bmatrix}_{jk} \qquad (3.8)$$

where we have denoted

$$C(\mathbf{r},t;\mathbf{r}',0) \equiv ict\{\pi[(\mathbf{r}-\mathbf{r}')^2-(ct-i\varepsilon)^2]\}^{-2} \qquad (3.9)$$

$$C^*(\mathbf{r},t;\mathbf{r}',0) \equiv -ict\{\pi[(\mathbf{r}-\mathbf{r}')^2-(ct+i\varepsilon)^2]\}^{-2} = C(\mathbf{r},-t;\mathbf{r}',0)$$

and the limit $\varepsilon \to 0$ is understood.
The last formula exhibits the transition probability amplitude that characterizes the **quantum Cauchy stochastic process**. To the best our knowledge, this is for the first time when this process appears. Earlier the quantum Brownian process , that corresponds to a massive particle, has been studied along with the quantum Poisson process and few other processes [21], [28], [29].

We note in passing that the photon's Hamiltonian $\mathrm{H} = c(\mathbf{S}\cdot\mathbf{p})$ depends on momentum, only. Hence the corresponding phase-space path integral for the transition amplitude can be performed(first over the coordinates $\mathbf{q}_k$ which leads to the product of $\delta(\mathbf{p}_k - \mathbf{p}_{k+1})$, the last momentum integral ,over the unpaired momentum $\mathbf{p}_N$ yields the above result, [32], [33]). The conservation of momentum ensures the known straight-line propagation of a photon.
Continuing analitically in time

$$t \to -it_E \qquad (t_E \text{ –Euclidean time})$$

we get the transition probability of the **classic Cauchy process** which belongs to a wider class of stochastic processes called **Le'vy processes** [34]. Both, Cauchy process and Brownian motion are members of this larger family of Markovian processes

$$C(\mathbf{r},-it_E;\mathbf{r}',0) = ct_E\{\pi[(\mathbf{r}-\mathbf{r}')^2+c^2t_E^2]\}^{-2}, \qquad t_E \geq 0, \quad \mathbf{r},\ \mathbf{r}' \subseteq R^3 \ (3.10)$$

Coming back to the matrix element (3.4), we find now using (3.9)

$$\langle\lambda|U(t)|\lambda'\rangle' = \tfrac{1}{2}[\langle\boldsymbol{\gamma}_\lambda|C|\boldsymbol{\gamma}_\lambda\rangle' + \langle\bar{\boldsymbol{\gamma}}_\lambda|C^*|\bar{\boldsymbol{\gamma}}_\lambda\rangle']\delta_{\lambda\lambda'} \qquad (3.11)$$

where we have denoted

$$\langle\boldsymbol{\gamma}_\lambda|C|\boldsymbol{\gamma}_{\lambda'}\rangle' \equiv \sum_{k,j=1}^{3}\int d^3r d^3r' \langle\boldsymbol{\gamma}_\lambda|\mathbf{r}\rangle_k C(\mathbf{r},t;\mathbf{r}',0)\delta_{kj}\langle\mathbf{r}'|\boldsymbol{\gamma}_{\lambda'}\rangle'_j \qquad (3.12)$$

and similarly for the anti-photon contribution. We also used the notation that is consistent with (2.38), taken at $t = 0$

$$\langle\mathbf{r}|1\rangle_j = \tfrac{1}{\sqrt{2}}\langle\mathbf{r}|\boldsymbol{\gamma}_1\rangle_j \qquad j = 1,2,3$$

$$\langle\mathbf{r}|1\rangle_j = \tfrac{1}{\sqrt{2}}\langle\mathbf{r}|\bar{\boldsymbol{\gamma}}_1\rangle_j \qquad j = 4,5,6 \qquad (3.13)$$

$$\langle\mathbf{r}|-1\rangle_j = \tfrac{1}{\sqrt{2}}\langle\mathbf{r}|\boldsymbol{\gamma}_{-1}\rangle_j \qquad j = 1,2,3$$

$$\langle\mathbf{r}|-1\rangle_j = \tfrac{1}{\sqrt{2}}\langle\mathbf{r}|\bar{\boldsymbol{\gamma}}_{-1}\rangle_j \qquad j = 4,5,6$$

Formula (3.11) shows that C(...) propagates photons while its complex conjugate $C^*(...)$ propagates anti-photons.

An arbitrary photon's state can be presented as the linear combination of states with definite helicities

$$|F\rangle = \alpha_F|1\rangle + \beta_F|-1\rangle \qquad (3.14)$$

where the coefficiets are

$$\alpha_F = (8\pi E)^{-1}\langle 1|F\rangle$$

$$\beta_F = (8\pi E)^{-1}\langle -1|F\rangle \qquad (3.15)$$

This allows to calculate the general matrix element

$$\langle F|U(t)|F'\rangle = \alpha_F^*\alpha_{F'}\langle 1|U(t)|1\rangle + \beta_F^*\beta_{F'}\langle -1|U(t)|-1\rangle \qquad (3.16)$$

We want to stress the fundamental difference of the photon's propagator as given in (3.9) and that for a massive spinless particle

$$B(\mathbf{r},t;\mathbf{r}',0) = (\tfrac{m}{2\pi i h t})^{3/2}\exp\{\tfrac{im(\mathbf{r}-\mathbf{r}')^2}{2ht}\}, \qquad t \geq 0, \qquad \mathbf{r},\ \mathbf{r}' \in R^3 \quad (3.17)$$

which corresponds to the quantum Brownian process [20], [21]. The massless photon propagator can not be obtained from Proca' massive vector particle theory by taking the mass parameter to zero [35], [36]. The limit is singular one as it involves blowing up terms proportionate to m^{-2}.

Thus the quantum Cauchy process complements the quantum Brownian motion process and should be treated as a stand alone feature of the stochastic formulation of the quantum mechanics.

4. CONCLUDING REMARKS

The transition amplitudes for photons and anti-photons are needed for the rigorous definition of Feynman integrals used in quantum mechanics and in quantum field theory [37], [38].

They might also help with with the temporal description of the double-slit experiment. What is missing, though, is the transition amplitude in the presence of absorbing or reflecting screen in which the slits are made. We have found the transition amplitude in the empty space, so far [39], [40].

One knows, [41], [42], [43], that the classical electrodynamics is sufficiently accurate when the number of photons in a field is large and the field is strong.

APPENDIX

We want to develop a practical formula for the evolution matrix $U(t)$ by expressing it through the helicity matrix. According to (1.8)

$$\mathsf{H} = -ic\hbar(\mathbf{S}\cdot\nabla) \equiv c\hbar\sqrt{-\triangle}\,\Lambda = c\hbar\begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \qquad (A.1)$$

We find from here the square of H

$$\mathsf{H}^2 = (c\hbar\sqrt{-\triangle}\,)^2\Lambda^2 = (c\hbar)^2\{[\partial_\lambda\partial_\rho] - \triangle\mathbf{1}\} \qquad (A.2)$$

where we have denoted

$$[\partial_\lambda\partial_\rho] \equiv \begin{bmatrix} \partial_x\partial_x & \partial_x\partial_y & \partial_x\partial_z \\ \partial_y\partial_x & \partial_y\partial_y & \partial_y\partial_z \\ \partial_z\partial_x & \partial_z\partial_y & \partial_z\partial_z \end{bmatrix} \qquad (A.3)$$

It is remarkable that

$$\mathsf{H}[\partial_\lambda\partial_\rho] = [\partial_\lambda\partial_\rho]\mathsf{H} = \mathbf{0} \qquad (A.4)$$

which leads to

$$H^3 = (c\hbar\sqrt{-\Delta})^3\Lambda \tag{A.5}$$

Further, using (A.4) again, we get

$$H^4 = H^3H = (c\hbar\sqrt{-\Delta})^4\Lambda^2 \tag{A.6}$$

In general, by the complete induction, we get

$$H^{2n} = (c\hbar\sqrt{-\Delta})^{2n}\Lambda^2 \qquad n = 1,2,\ldots \tag{A.7}$$

$$H^{2n+1} = (c\hbar\sqrt{-\Delta})^{2n+1}\Lambda \qquad n = 0,1,\ldots \tag{A.8}$$

This allows to further simplify the evolution matrix

$$U(t) = \exp\{-\tfrac{i}{\hbar}tH\} = \mathbf{1} - \Lambda^2 + \cos(ct\sqrt{-\Delta})\Lambda^2 - i\sin(ct\sqrt{-\Delta})\Lambda \backsim$$

$$\backsim \exp\{-i\lambda ct\sqrt{-\Delta}\}\mathbf{1} \tag{A.9}$$

when acting on a state of helicity $\lambda = \pm 1$. Thus the formula (3.7) holds.
To help with the formula (3.8), notice that with $\xi \equiv |\mathbf{r} - \mathbf{r}'|$, $\eta \equiv c\lambda t$, we get after the integration over the spherical angles

$$C(\mathbf{r},\lambda\mathbf{t};\mathbf{r}',0) = (4\pi^2\hbar\xi)^{-1}\partial_\eta\int_o^\infty dp[\exp\{\tfrac{i}{\hbar}p(\xi - \eta + i\varepsilon)\} - \exp\{-\tfrac{i}{\hbar}p(\xi + \eta - i\varepsilon)\}]$$

where $\varepsilon \to 0$. $\qquad (A.10)$

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