

Dynamics of Heterogeneous Voter Models

Sid Redner (physics.bu.edu/~redner)

Nonlinear Dynamics of Networks UMD April 5-9, 2010

T. Antal (BU→Harvard), N. Gibert (ENSTA), N. Masuda (Tokyo),
M. Mobilia (BU→Leeds), V. Sood (BU→NBI), D. Volovik (BU)

NSF DMR0535503 & DMR0906504

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The classic voter model

3 basic results

Voting on complex networks

T. Antal, V. Sood

new conservation law

two time-scale route to consensus

short consensus time

Strategic voting (>2 states)

M. Mobilia, D. Volovik

long time-scale switching

Partisan voting

N. Masuda, N. Gibert

selfishness vs. collectiveness

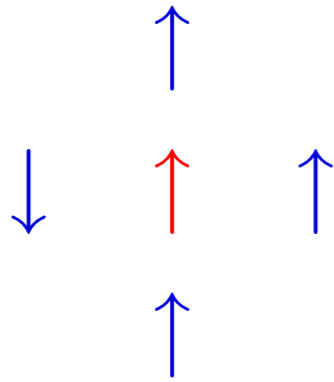
ultraslow evolution

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Classic Voter Model

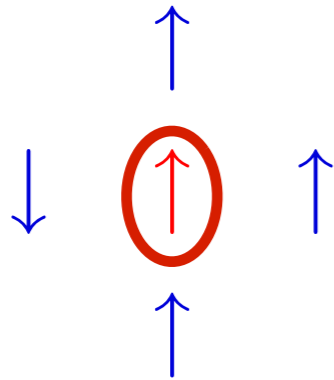
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0. Binary voter variable at each site i

Classic Voter Model

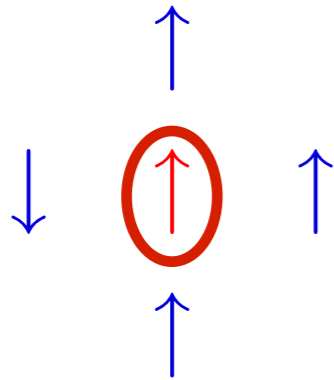
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1. Pick a random voter

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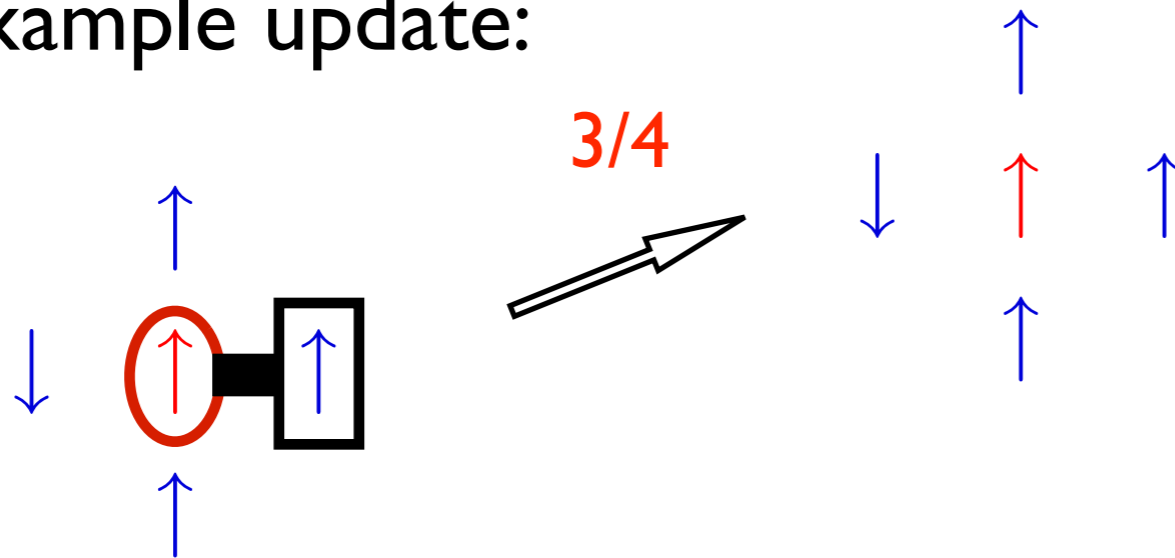
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2. Assume state of randomly-selected neighbor
individual has no self-confidence & adopts neighbor's state

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Example update:



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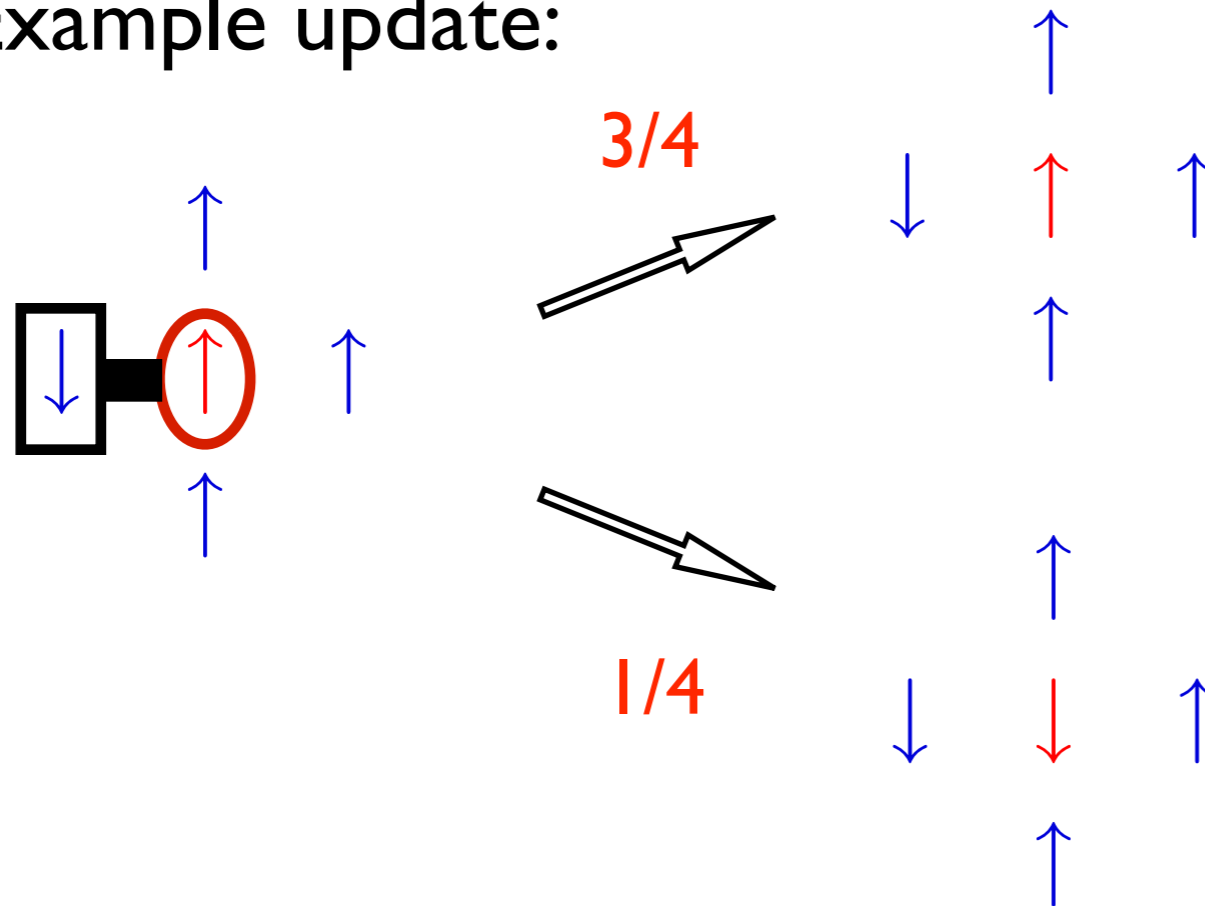
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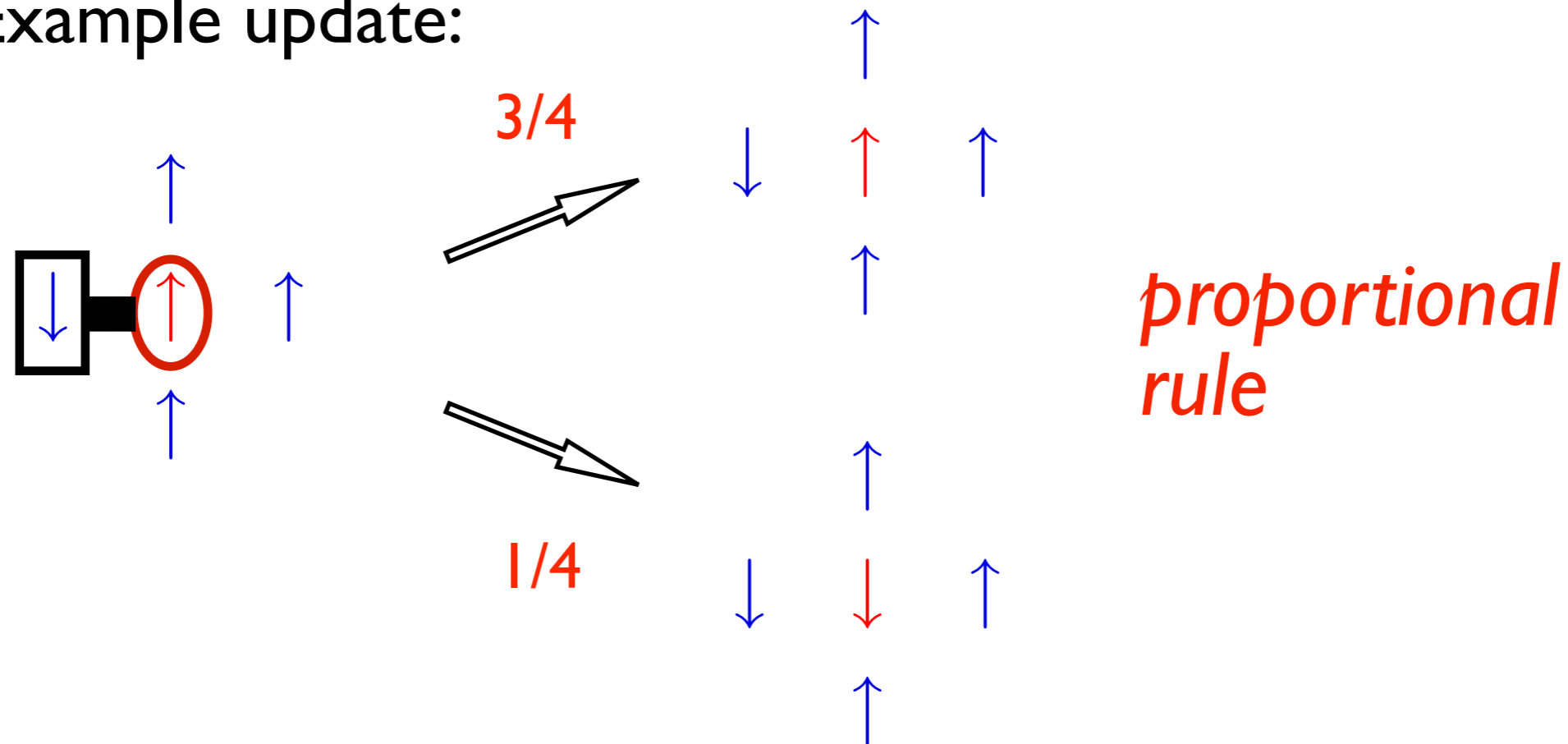
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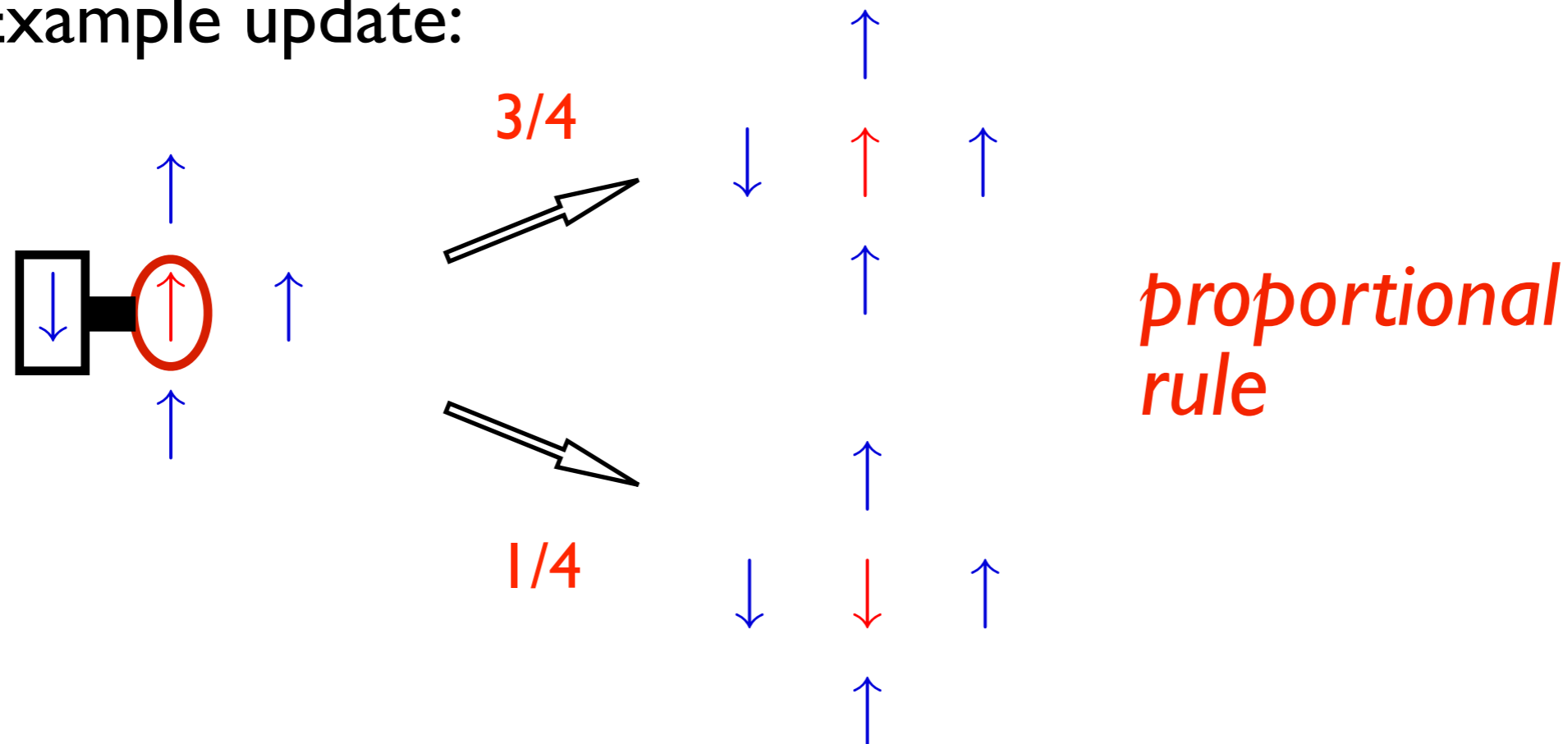
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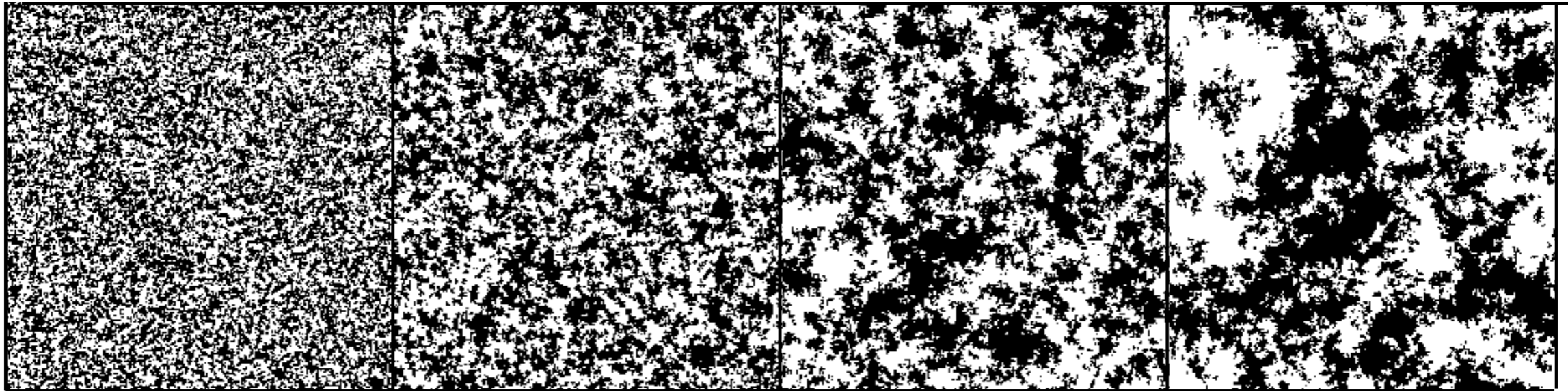


0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
individual has no self-confidence & adopts neighbor's state
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

Voter Model Evolution

Dornic et al. (2001)

random initial condition:



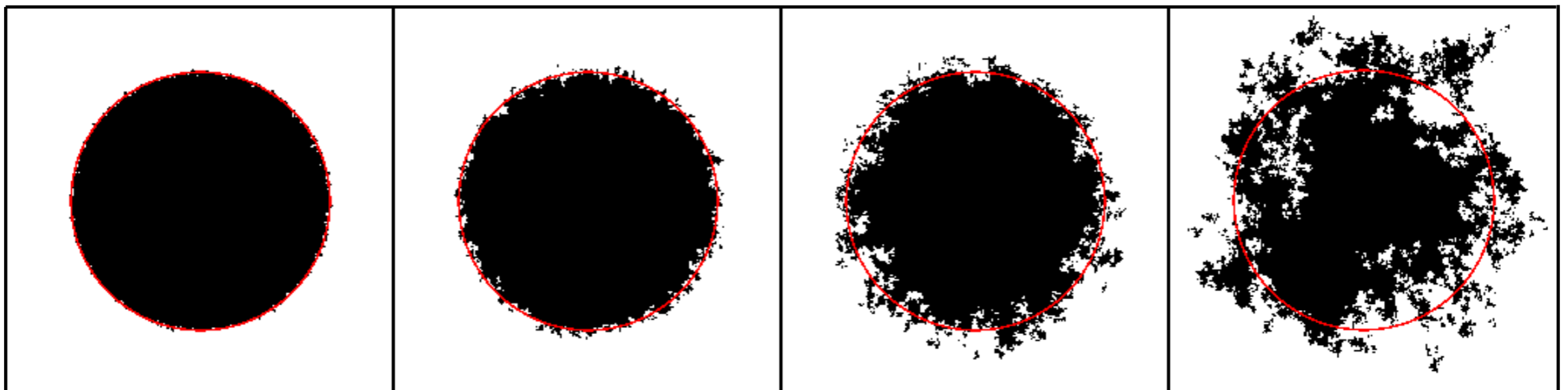
$t=4$

$t=16$

$t=64$

$t=256$

droplet initial condition:



Lattice Voter Model: 3 Basic Properties

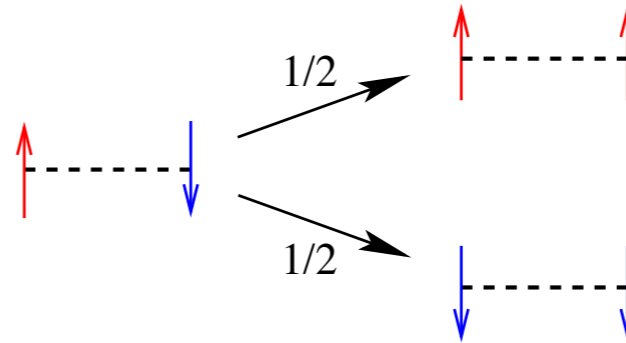
Lattice Voter Model: 3 Basic Properties

I. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

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Evolution of a single active link:

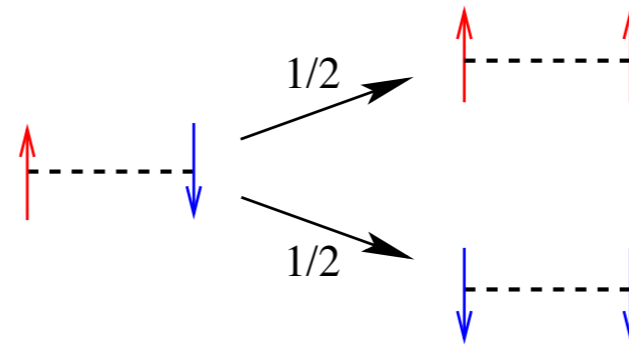


average
magnetization
conserved

Lattice Voter Model: 3 Basic Properties

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Evolution of a single active link:

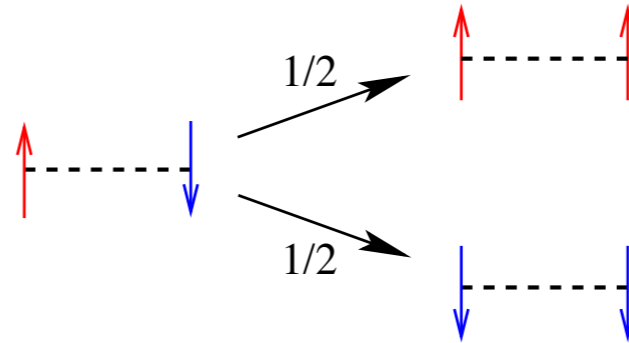


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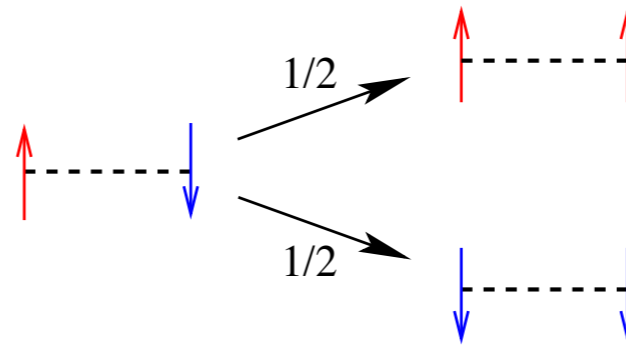
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2. Two-Spin Correlations

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:



average magnetization conserved

2. Two-Spin Correlations

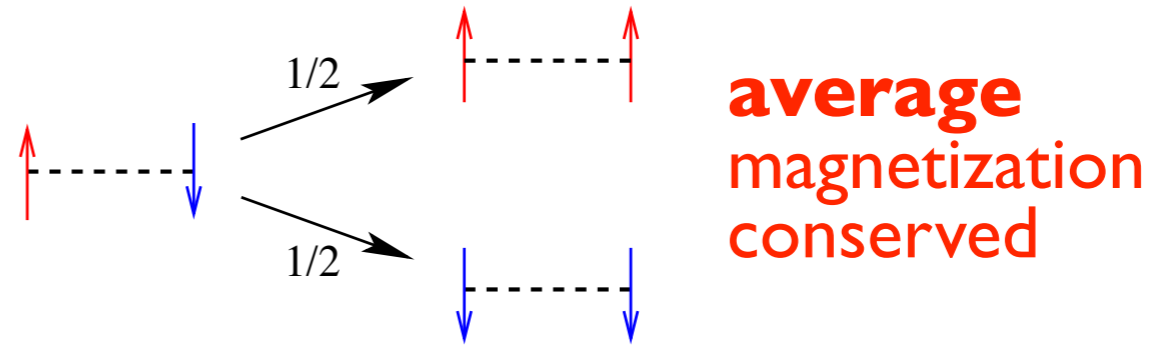
$$\frac{\partial c_2(\mathbf{r}, t)}{\partial t} = \nabla^2 c_2(\mathbf{r}, t)$$

$$\begin{aligned} c_2(r=0, t) &= 1 \\ c_2(r > 0, t=0) &= 0 \end{aligned}$$

Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

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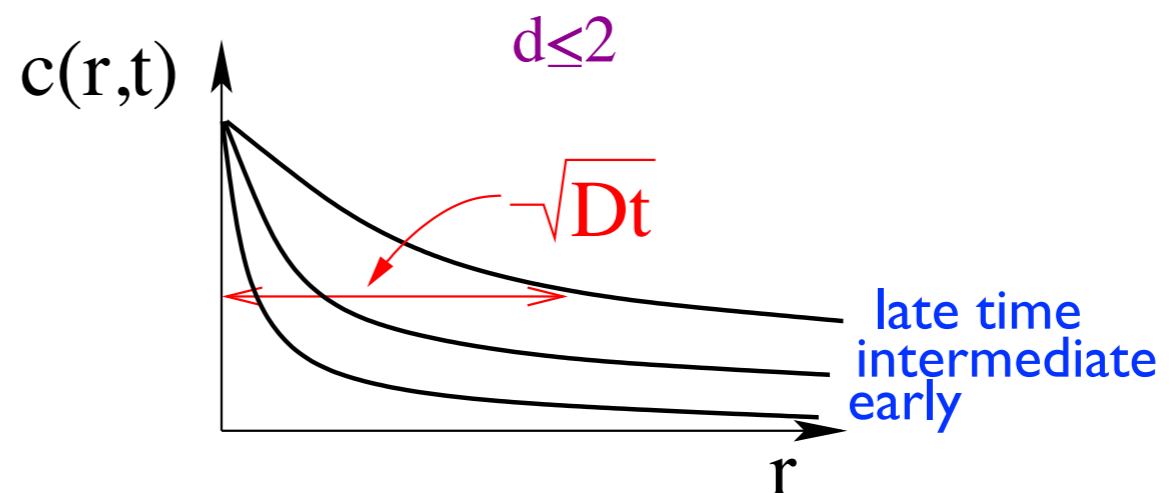
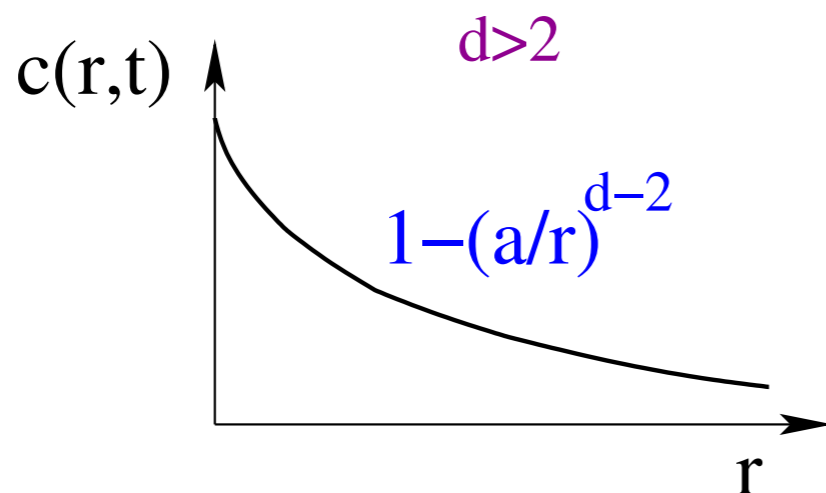


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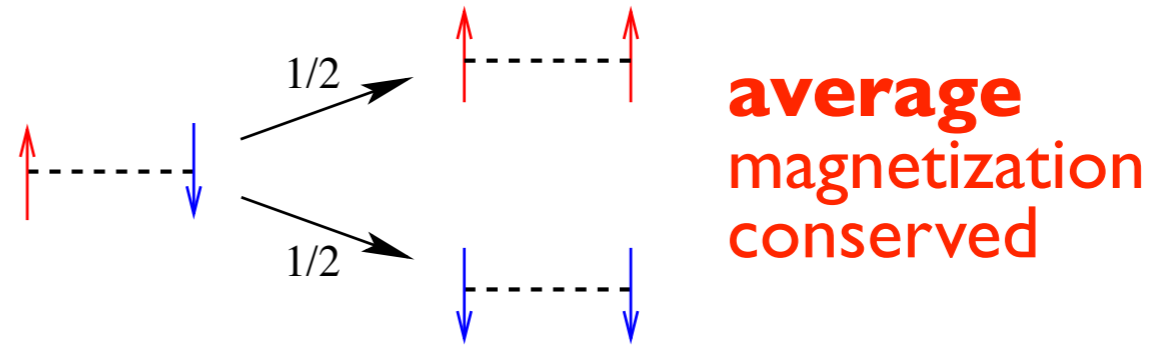
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Lattice Voter Model: 3 Basic Properties

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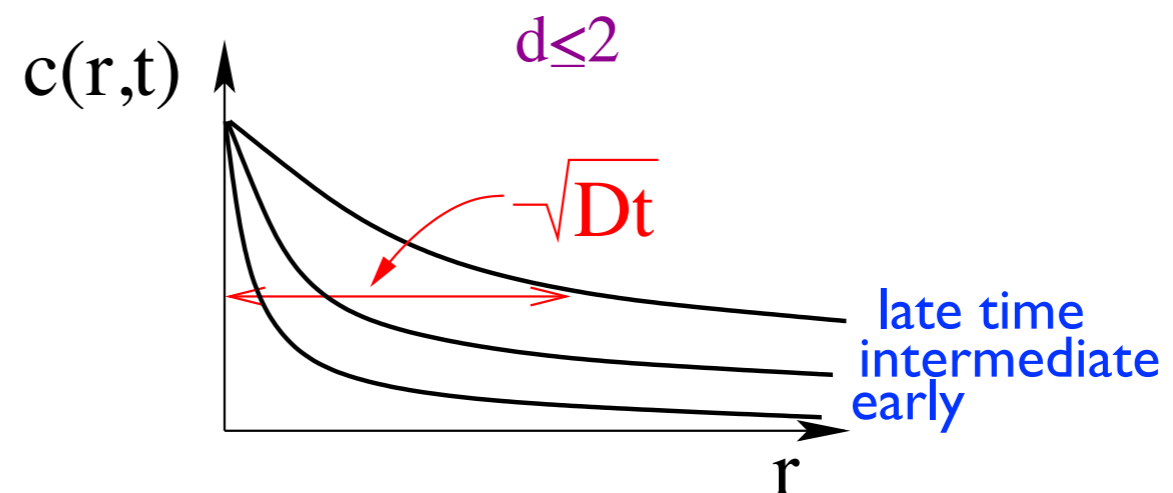
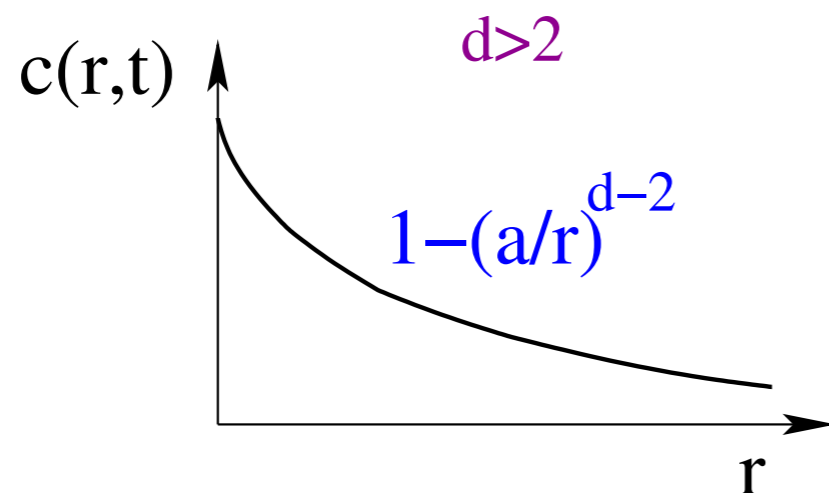


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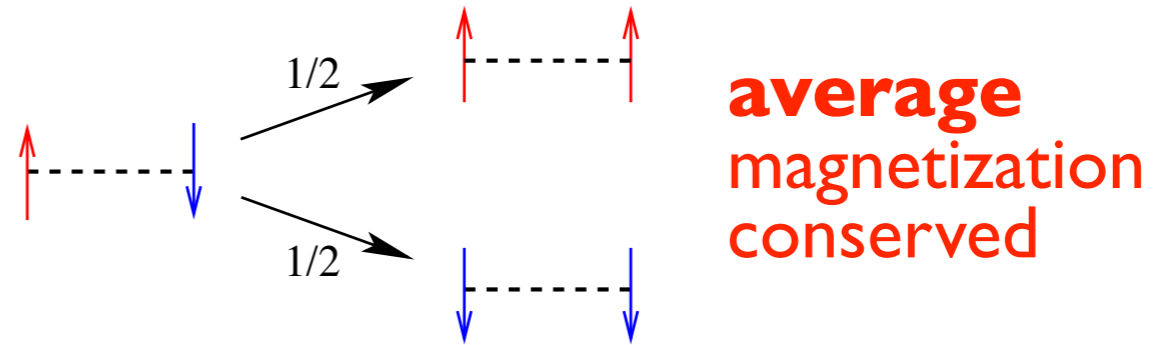


3. Consensus Time

Lattice Voter Model: 3 Basic Properties

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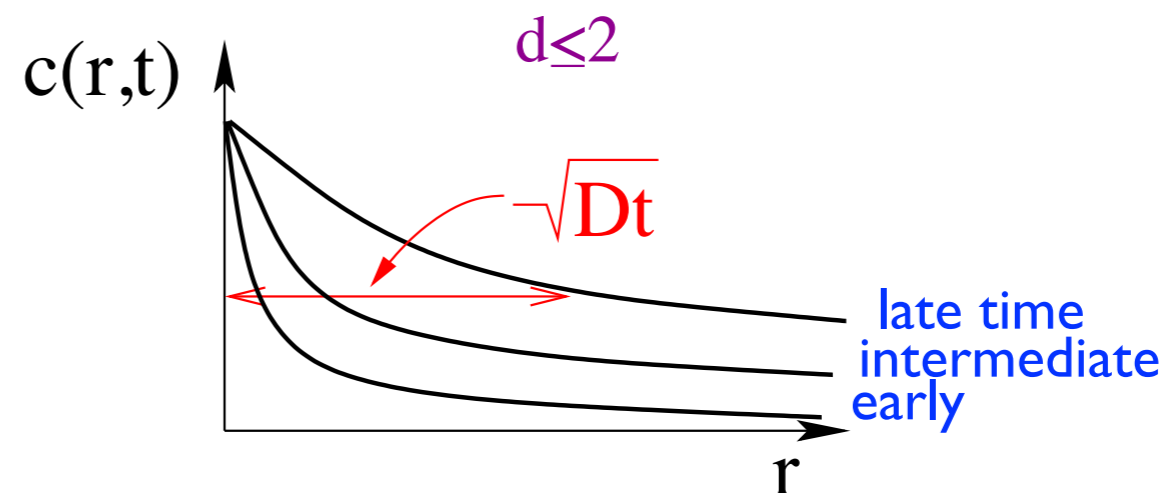
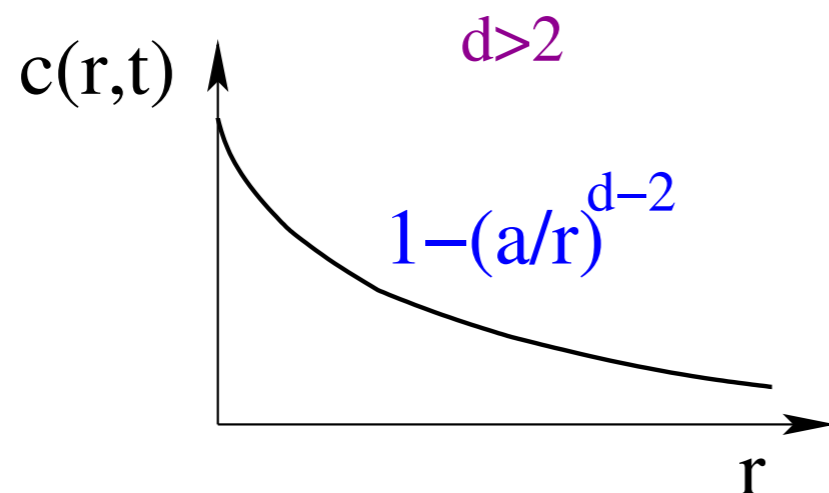


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3. Consensus Time

dimension	consensus time
1	N^2
2	$N \ln N$
> 2	N

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)

K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)

V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

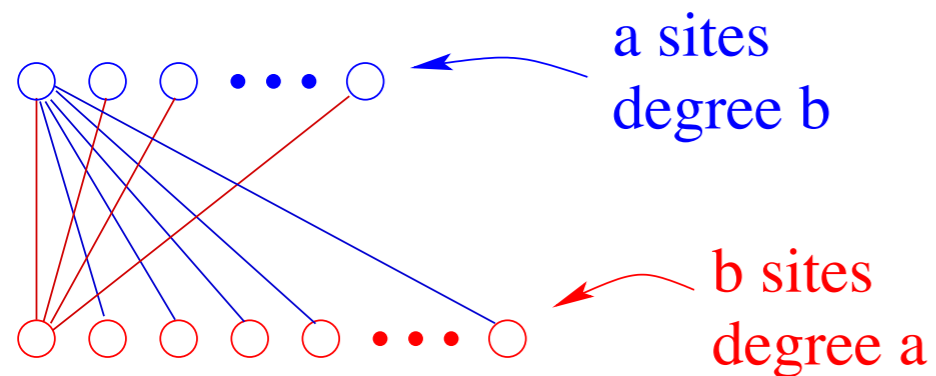
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illustrative example:
complete bipartite graph



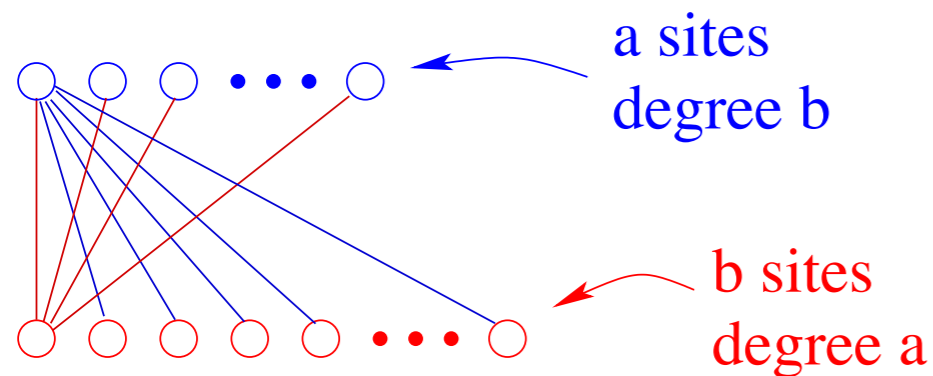
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$$dN_a = \frac{a}{a+b} \left[\frac{a - N_a}{a} \frac{N_b}{b} - \frac{N_a}{a} \frac{b - N_b}{b} \right]$$
$$dN_b = \frac{b}{a+b} \left[\frac{b - N_b}{b} \frac{N_a}{a} - \frac{N_b}{b} \frac{a - N_a}{a} \right]$$

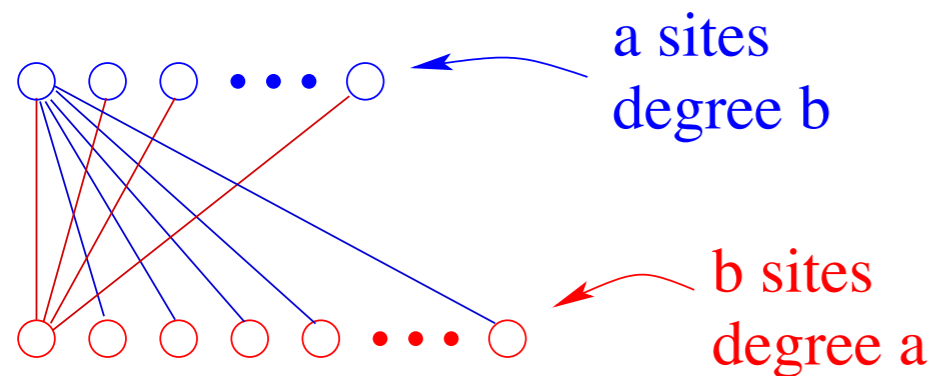
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illustrative example:
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pick site on
a sublattice



$$dN_a = \frac{a}{a+b} \begin{bmatrix} \frac{a - N_a}{a} & \frac{N_b}{b} \\ \frac{N_a}{a} & \frac{b - N_b}{b} \end{bmatrix}$$
$$dN_b = \frac{b}{a+b} \begin{bmatrix} \frac{b - N_b}{b} & \frac{N_a}{a} \\ \frac{N_b}{b} & \frac{a - N_a}{a} \end{bmatrix}$$

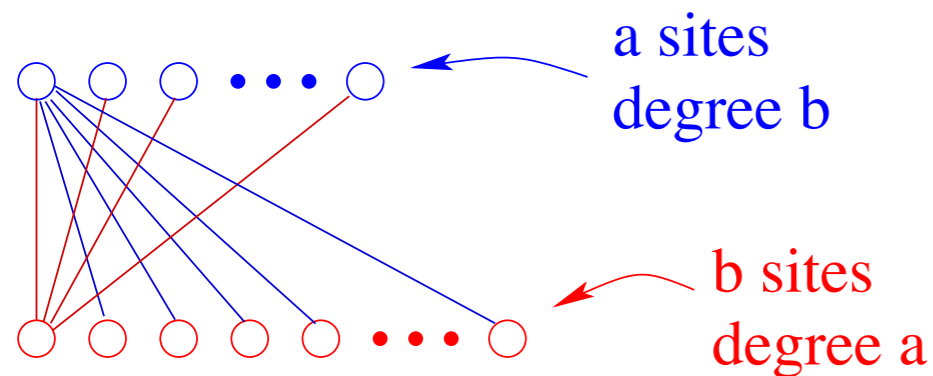
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$$\begin{aligned}
 dN_a &= \frac{a}{a+b} \begin{bmatrix} a - N_a & N_b \\ a & b \end{bmatrix} - \frac{N_a}{a} \frac{b - N_b}{b} \\
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pick site on a sublattice \downarrow pick \downarrow on a \downarrow

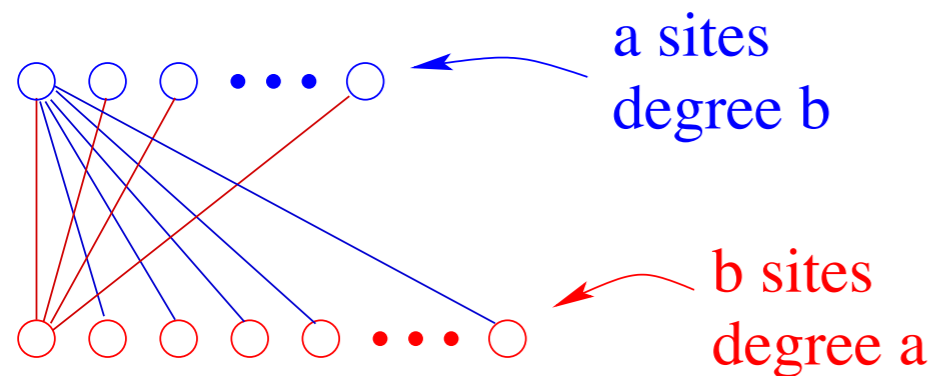
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 dN_b &= \frac{b}{a+b} \begin{bmatrix} \overset{\substack{\text{pick } \downarrow \\ \text{on a}}}{b} - N_b & N_a \\ \frac{b}{b} & \frac{N_a}{a} \end{bmatrix} - \frac{N_b}{b} \frac{a - N_a}{a}
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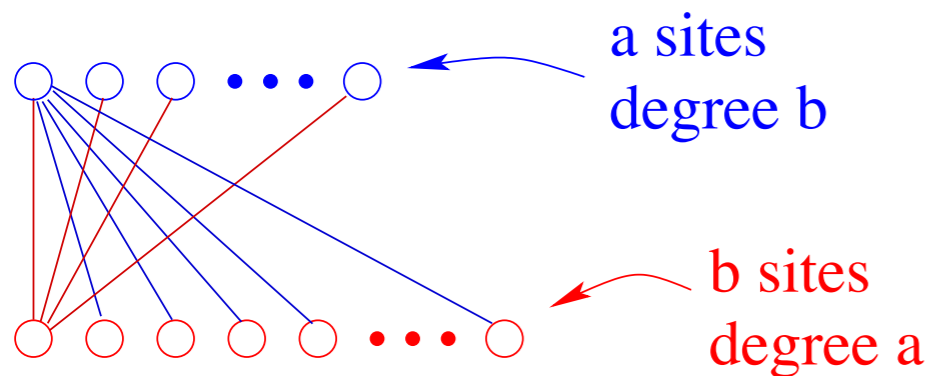
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 \end{aligned}$$

pick site on a sublattice

pick ↓ on a

pick ↑ on b sublattice

Subgraph densities: $\rho_a = N_a/a$, $\rho_b = N_b/b$ $dt = 1/(a+b)$

$$\begin{aligned}
 \rho_{a,b}(t) &= \frac{1}{2} [\rho_{a,b}(0) - \rho_{b,a}(0)] e^{-2t} + \frac{1}{2} [\rho_a(0) + \rho_b(0)] \\
 &\rightarrow \frac{1}{2} [\rho_a(0) + \rho_b(0)]
 \end{aligned}$$

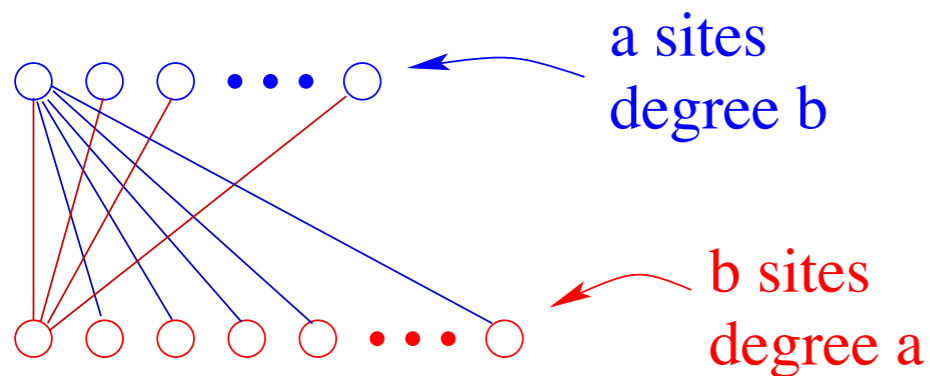
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pick site on a sublattice
pick ↓ on a
pick ↑ on b sublattice

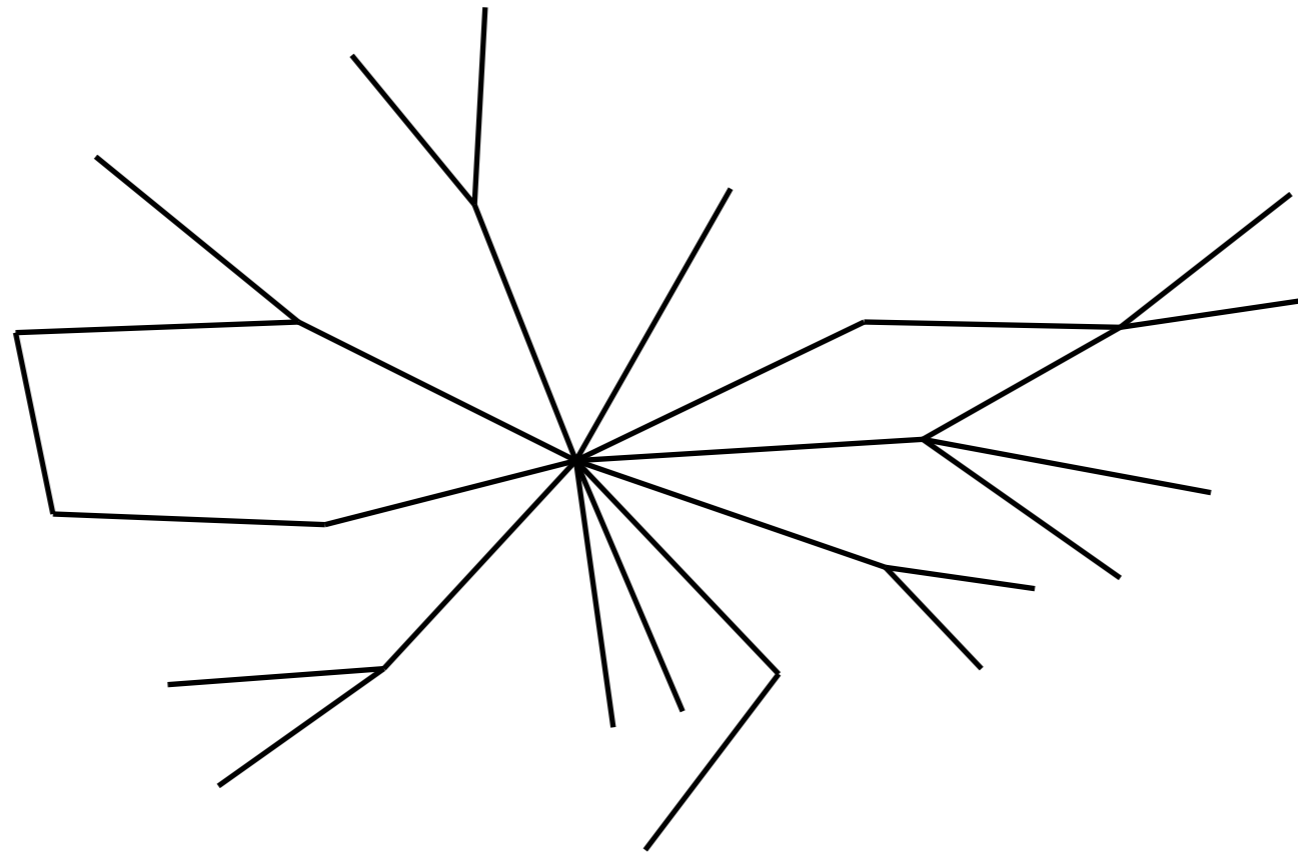
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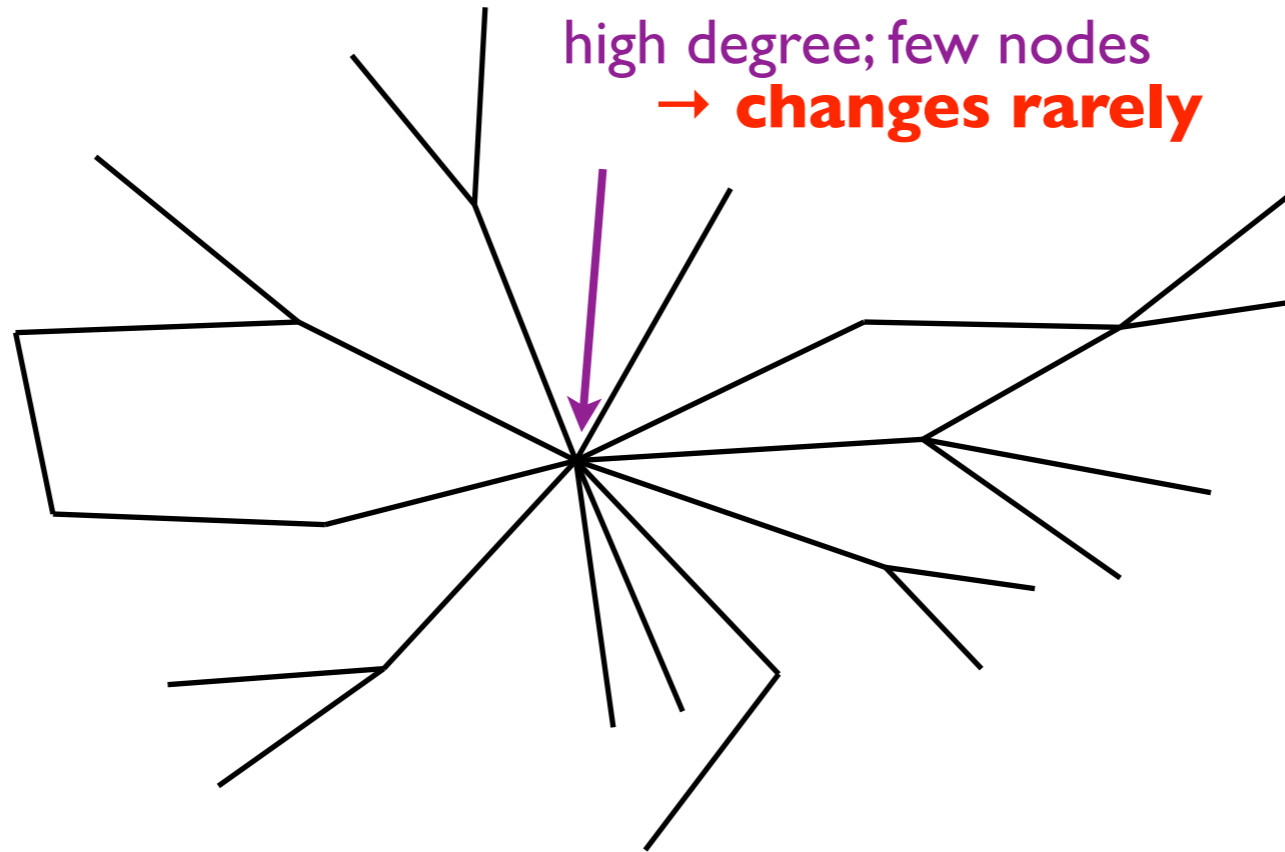
$$\rightarrow \frac{1}{2} [\rho_a(0) + \rho_b(0)] \quad \text{magnetization **not** conserved}$$

Voter Model on Complex Networks

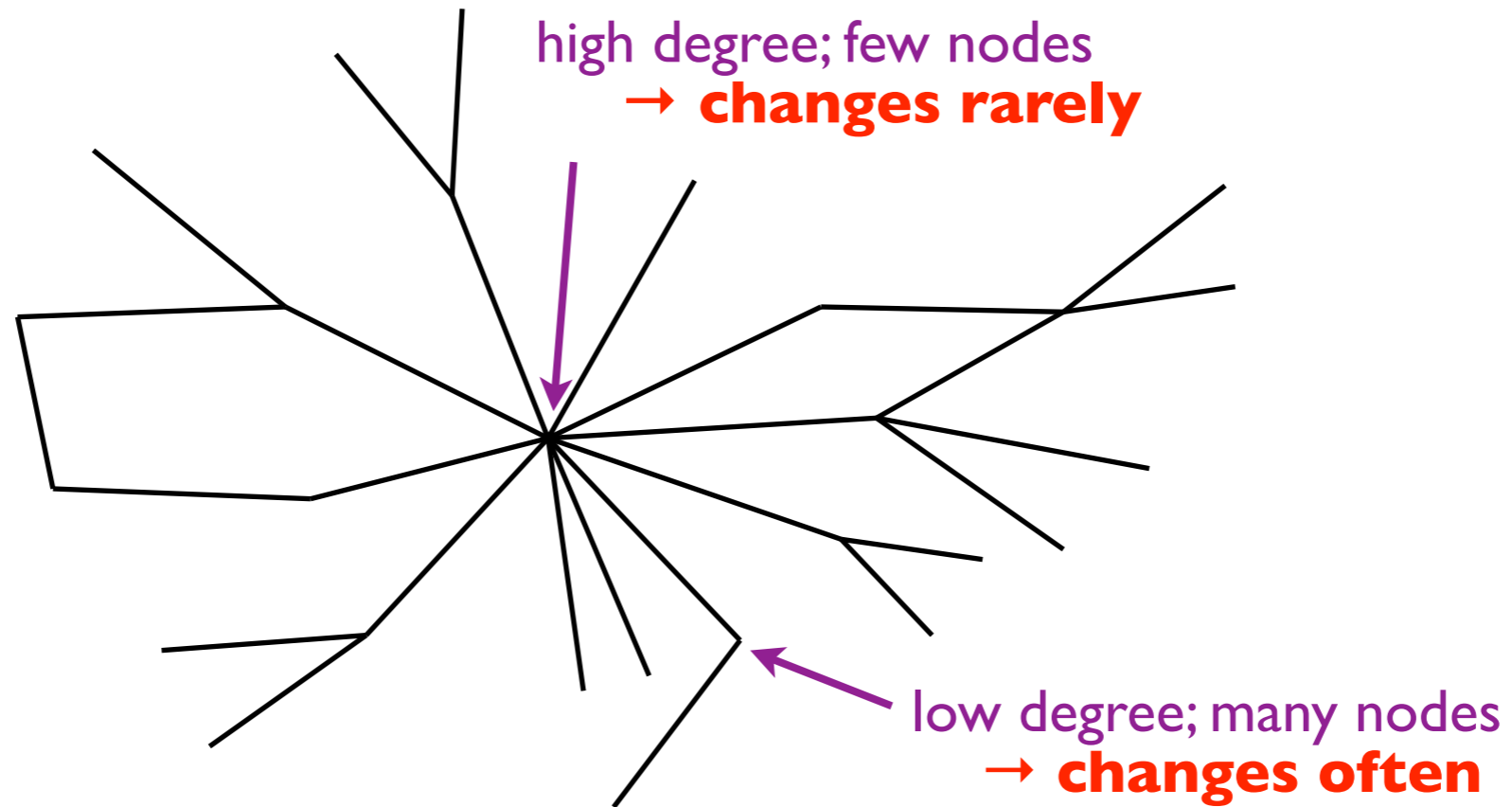
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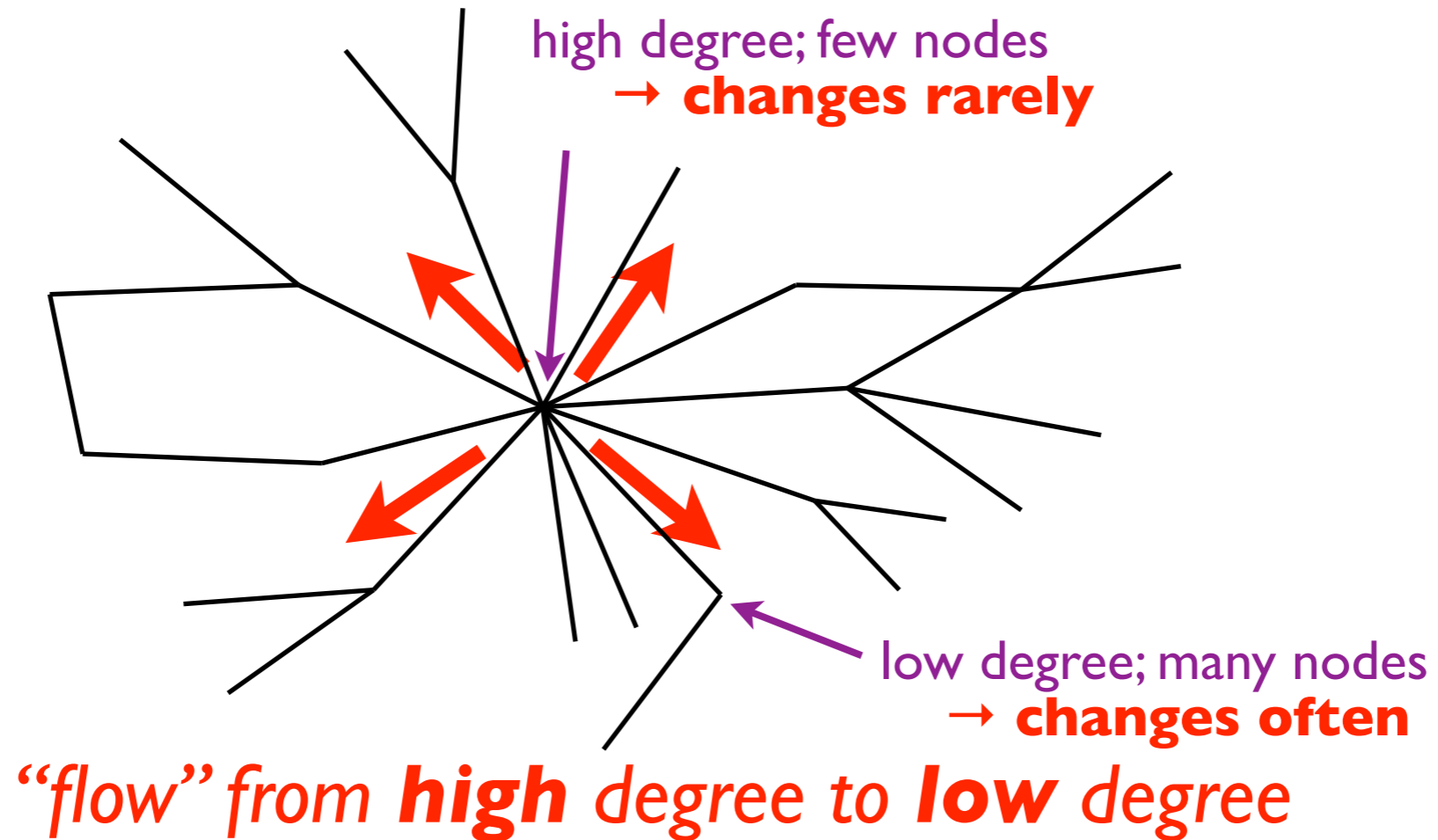
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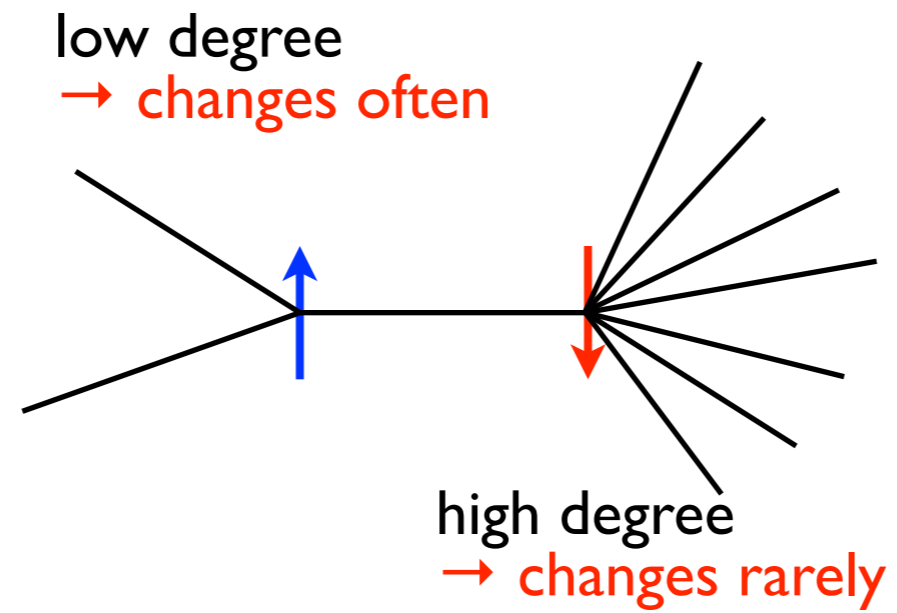
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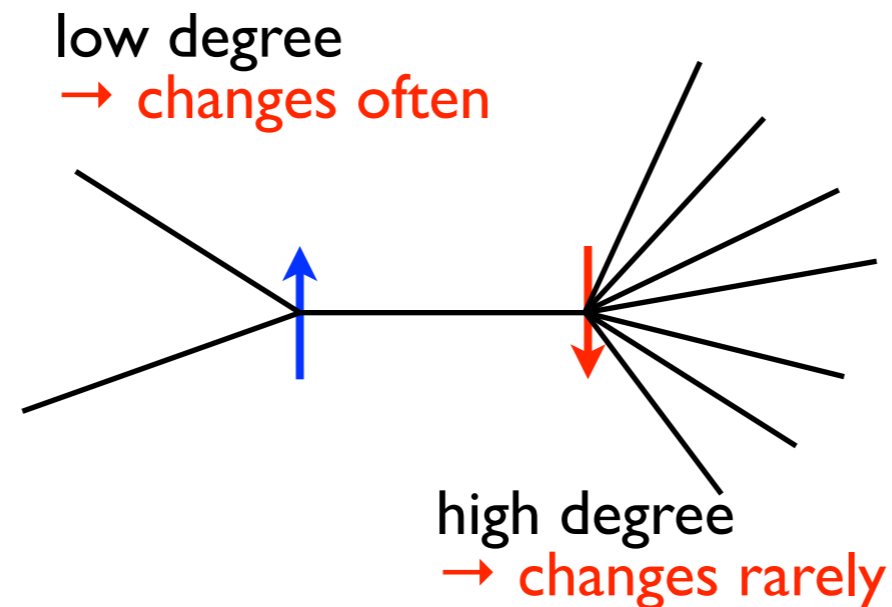
Voter Model on Complex Networks



New Conservation Law



New Conservation Law



to compensate different rates, consider:

degree-weighted
1st moment:

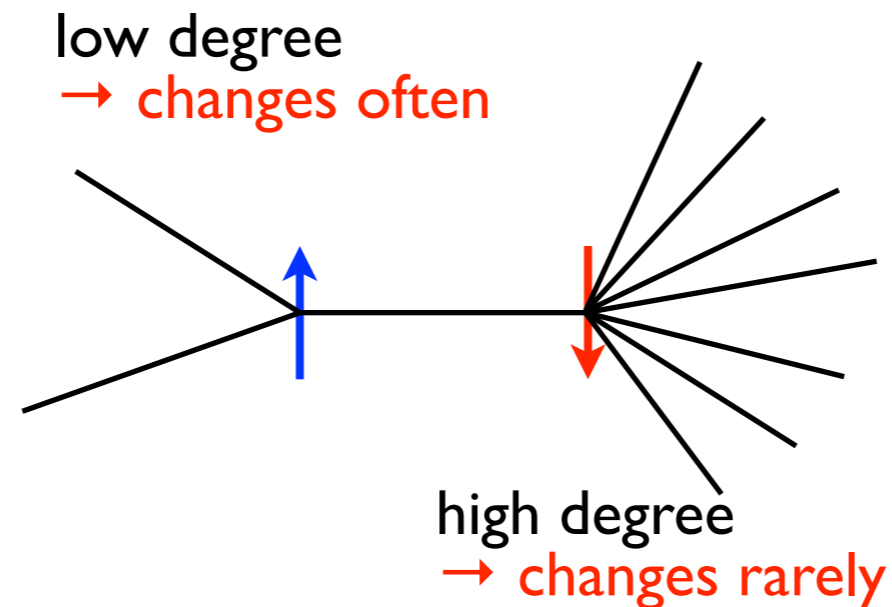
$$\omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k$$

μ_1 = av. degree

n_k = frac. nodes of degree k

ρ_k = frac. \uparrow on nodes of degree k

New Conservation Law



to compensate different rates, consider:

degree-weighted
1st moment:

$$\omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k \quad \text{conserved!}$$

μ_1 = av. degree

n_k = frac. nodes of degree k

ρ_k = frac. \uparrow on nodes of degree k

Exit Probability on Complex Graphs

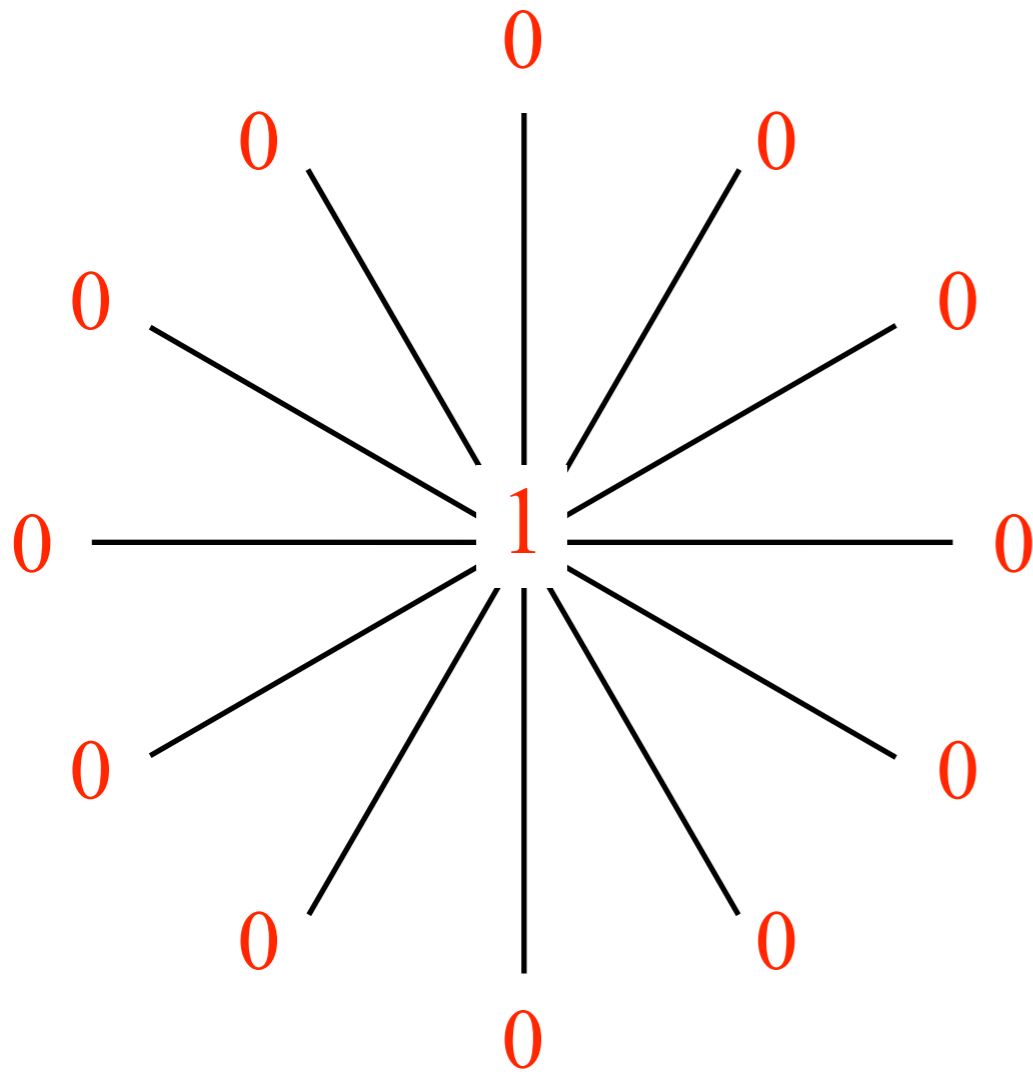
$$\mathcal{E}(\omega) = \omega$$

Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Extreme case: star graph

N nodes: degree 1
1 node: degree N

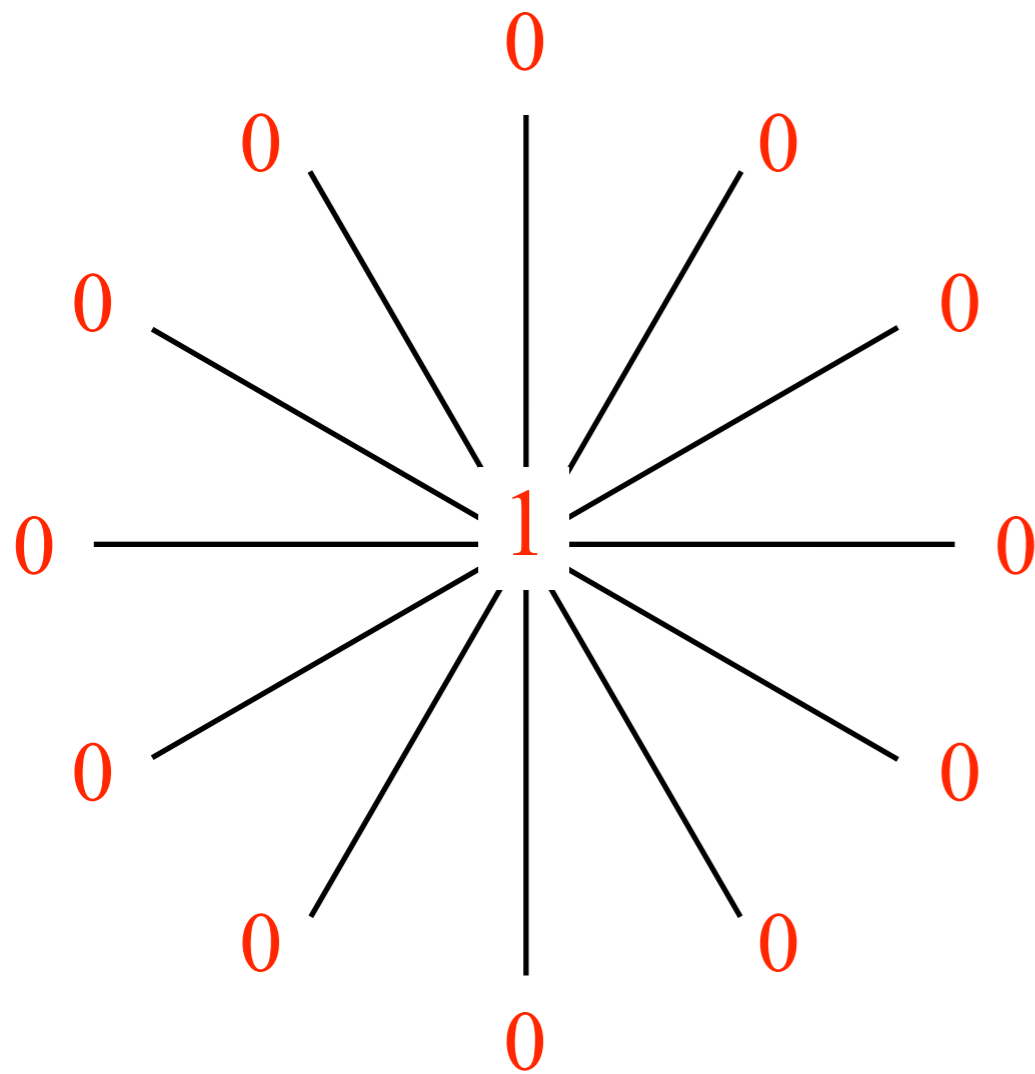


Exit Probability on Complex Graphs

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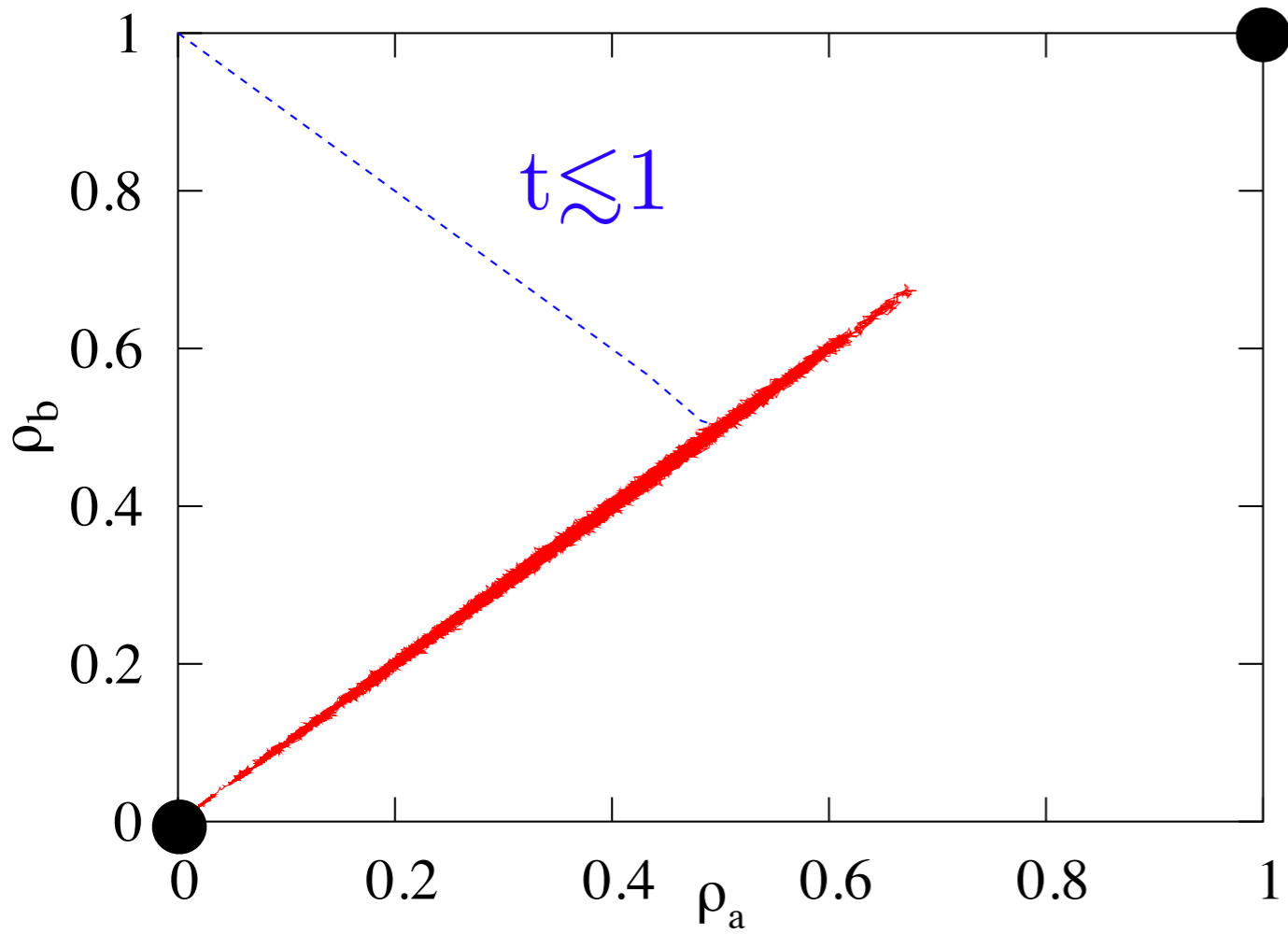


$$\omega = \frac{1}{\mu_1} \sum_k k n_k \rho_k = \frac{1}{2}$$

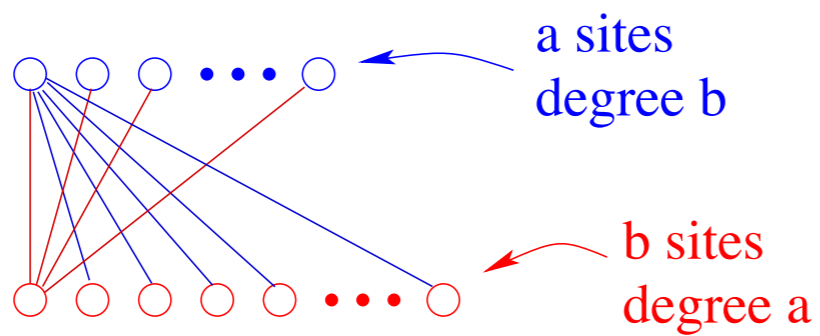
Final state: all 1 with prob. 1/2!

Route to Consensus on Complex Graphs

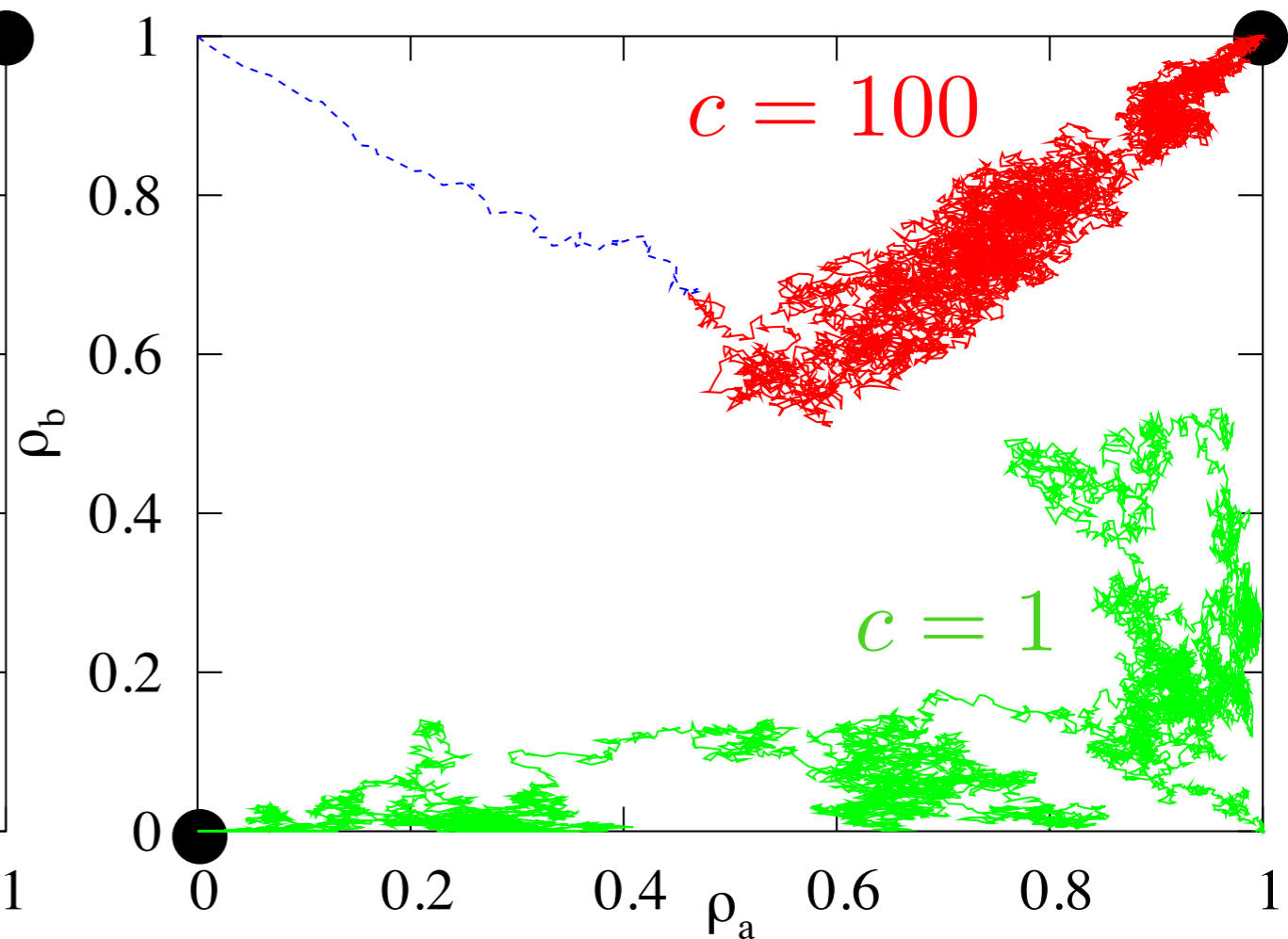
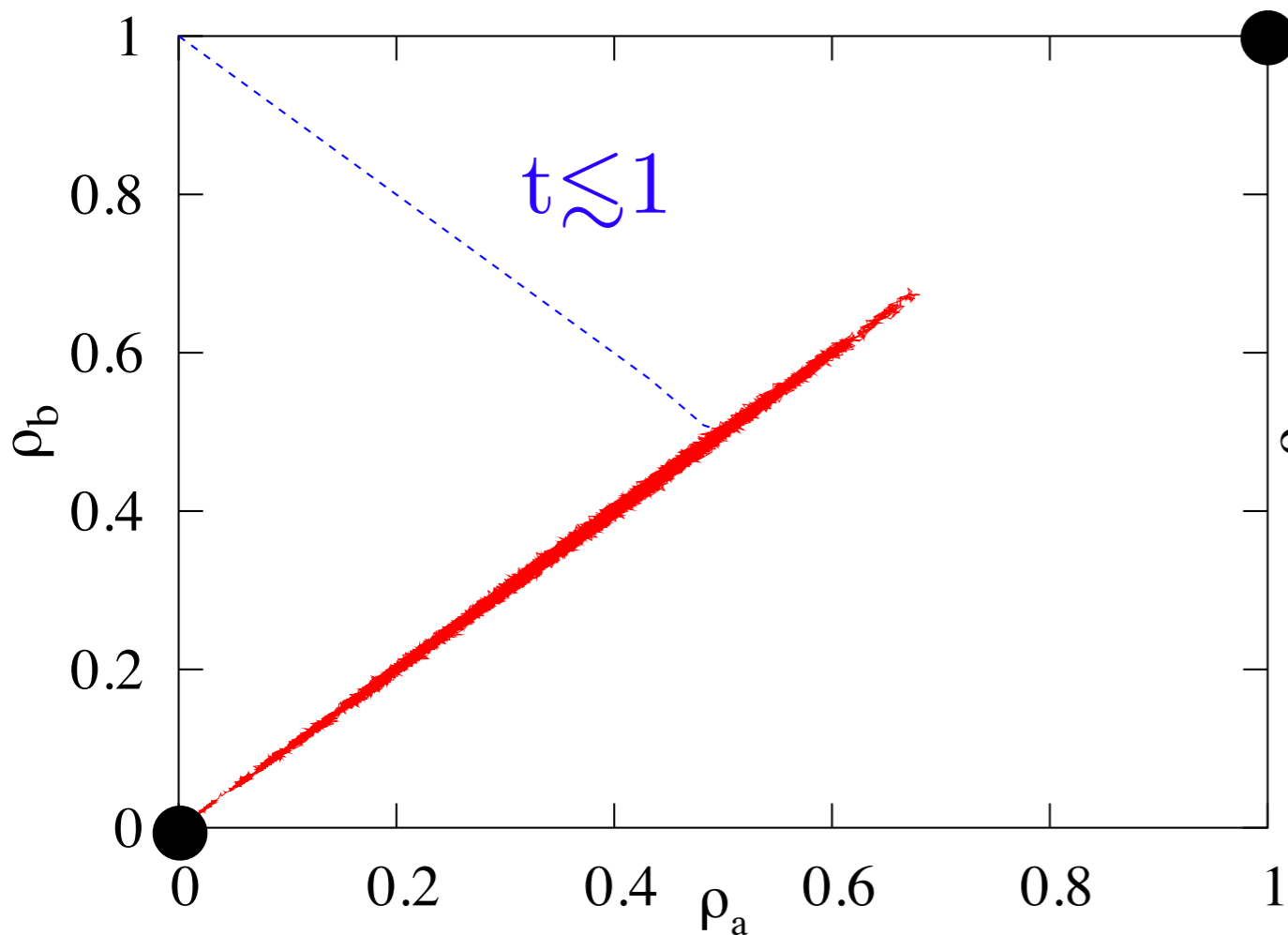
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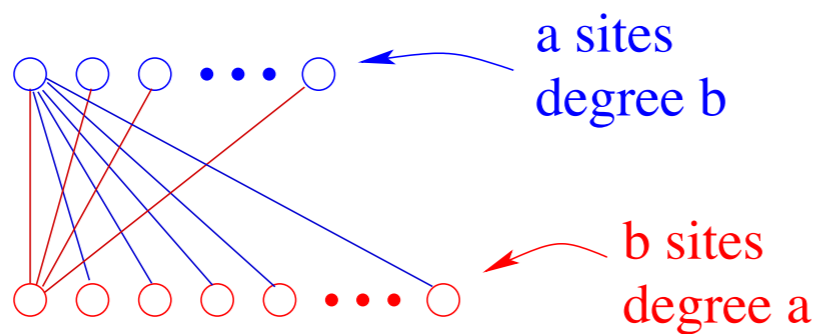
complete bipartite graph



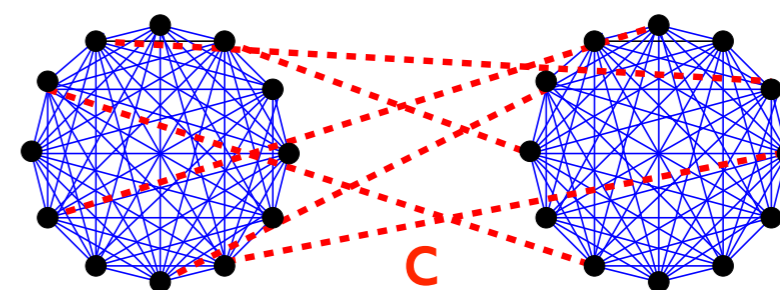
Route to Consensus on Complex Graphs



complete bipartite graph



two-clique graph



$N=10000$, C links/node

Consensus Time Evolution Equation

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

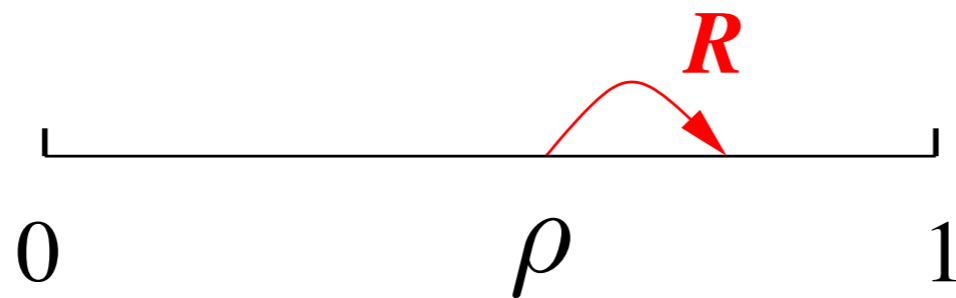
$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho + d\rho) + dt] \\ &\quad + \mathcal{L}(\rho)[T(\rho - d\rho) + dt] \\ &\quad + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt] \end{aligned}$$

Consensus Time Evolution Equation

warmup: complete graph

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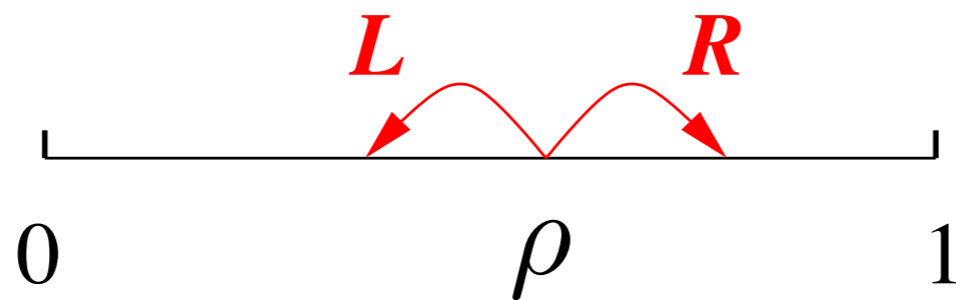
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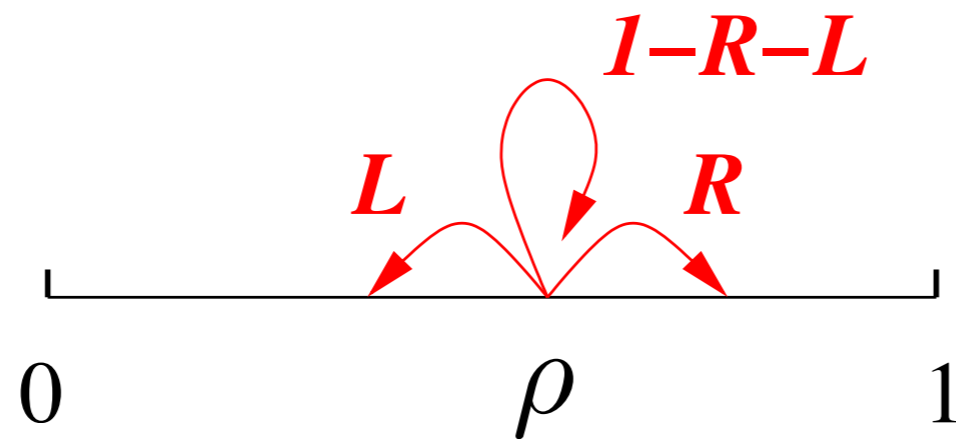
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$$\mathcal{R}(\rho) \equiv \text{prob}(\downarrow\uparrow \rightarrow \uparrow\uparrow)$$

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$$= \rho(1 - \rho)$$

Consensus Time on Complete Graph

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continuum limit:

$$T'' = -\frac{N}{\rho(1 - \rho)}$$

Consensus Time on Complete Graph

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continuum limit:

$$T'' = -\frac{N}{\rho(1-\rho)}$$

solution:

$$T(\rho) = -N [\rho \ln \rho + (1 - \rho) \ln(1 - \rho)]$$

Consensus Time on Heterogeneous Networks

$T(\{\rho_k\}) \equiv$ av. consensus time starting with density ρ_k
on nodes of degree k

$$\begin{aligned} T(\{\rho_k\}) &= \sum_k \mathcal{R}_k(\{\rho_k\}) [T(\{\rho_k^+\}) + dt] \\ &+ \sum_k \mathcal{L}_k(\{\rho_k\}) [T(\{\rho_k^-\}) + dt] \\ &+ \left[1 - \sum_k [\mathcal{R}_k(\{\rho_k\}) + \mathcal{L}_k(\{\rho_k\})] \right] [T(\{\rho_k\}) + dt] \end{aligned}$$

$$\begin{aligned} \mathcal{R}_k(\{\rho_k\}) &= \text{prob}(\rho_k \rightarrow \rho_k^+) & \mathcal{L}_k(\{\rho_k\}) &= n_k \rho_k (1 - \omega) \\ &= \frac{1}{N} \sum_x' \frac{1}{k_x} \sum_y P(\downarrow, \text{---}, \uparrow) \\ &= n_k \omega (1 - \rho_k) \end{aligned}$$

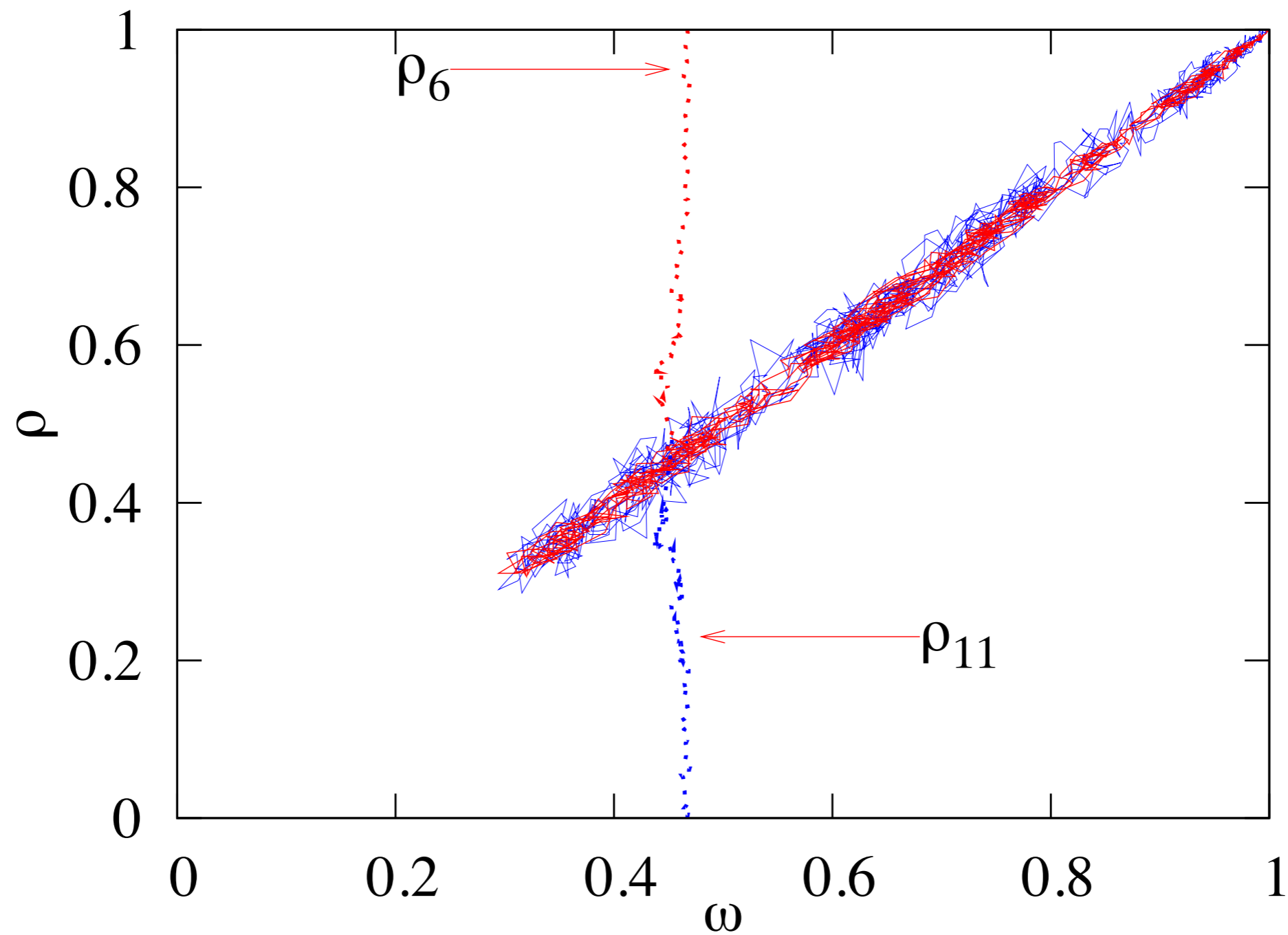
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

Molloy-Reed Scale-Free Network

$$n_k \sim k^{-2.5}, \quad \mu_1 = 8$$



Consensus Time on Heterogeneous Networks

continuum limit:

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now use $\rho_k \rightarrow \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega}$

Consensus Time on Heterogeneous Networks

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to give $\frac{\partial^2 T}{\partial \omega^2} = -\frac{N\mu_1^2/\mu_2}{\omega(1-\omega)}$

Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k) \frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k} \frac{\partial^2 T}{\partial \rho_k^2} \right] = -1$$

now use $\rho_k \rightarrow \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k} \frac{\partial}{\partial \omega} = \frac{kn_k}{\mu_1} \frac{\partial}{\partial \omega}$

to give $\frac{\partial^2 T}{\partial \omega^2} = -\frac{N\mu_1^2/\mu_2}{\omega(1-\omega)}$ same as $T'' = -\frac{N}{\rho(1-\rho)}$

with effective size $N_{\text{eff}} = N\mu_1^2/\mu_2$

Consensus Time for Power-Law Degree

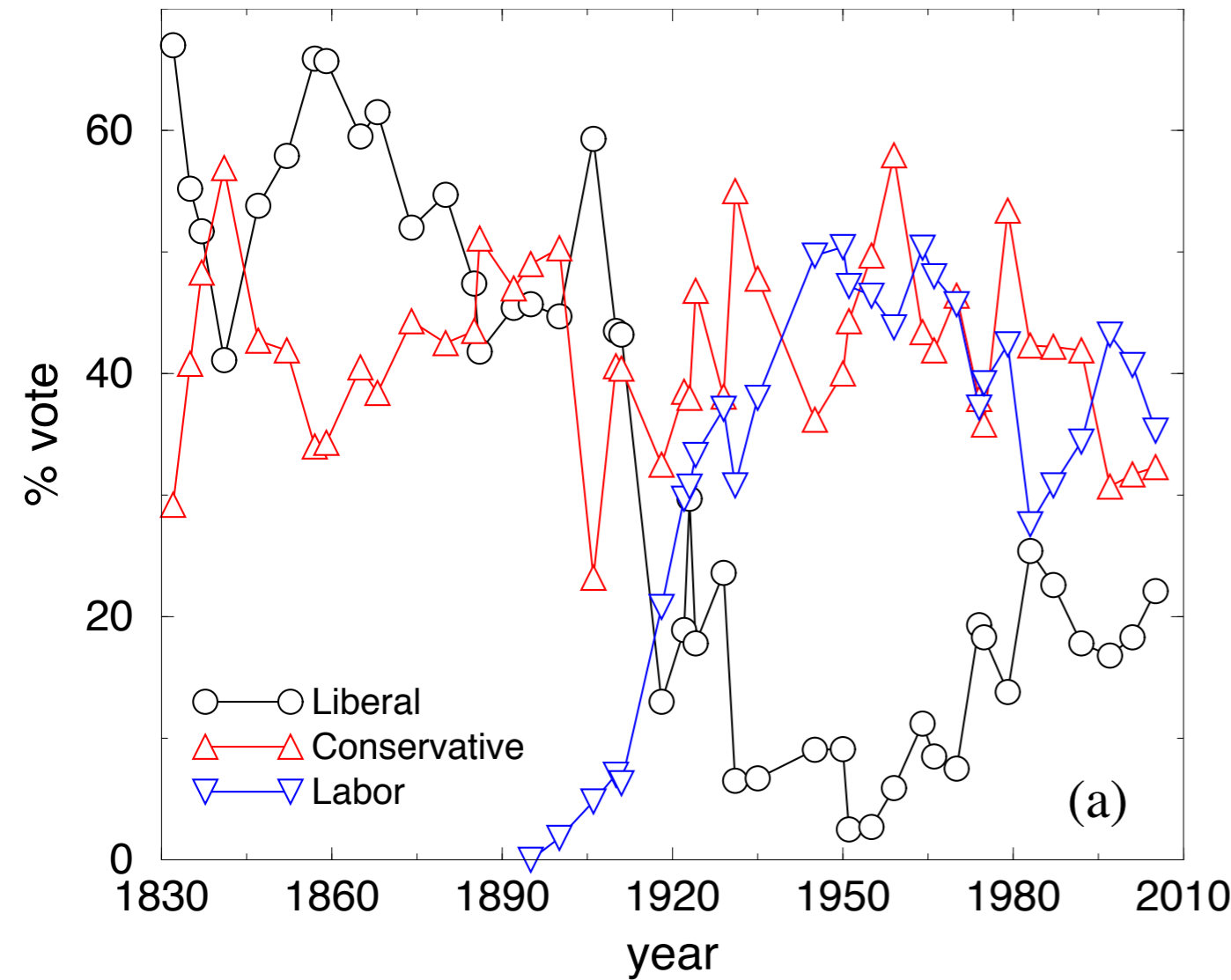
Distribution $n_k \sim k^{-\nu}$

$$T_N \propto N_{\text{eff}} = N \frac{\mu_1^2}{\mu_2} \sim \left\{ \begin{array}{ll} N & \nu > 3 \\ N / \ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{array} \right. \Bigg]$$

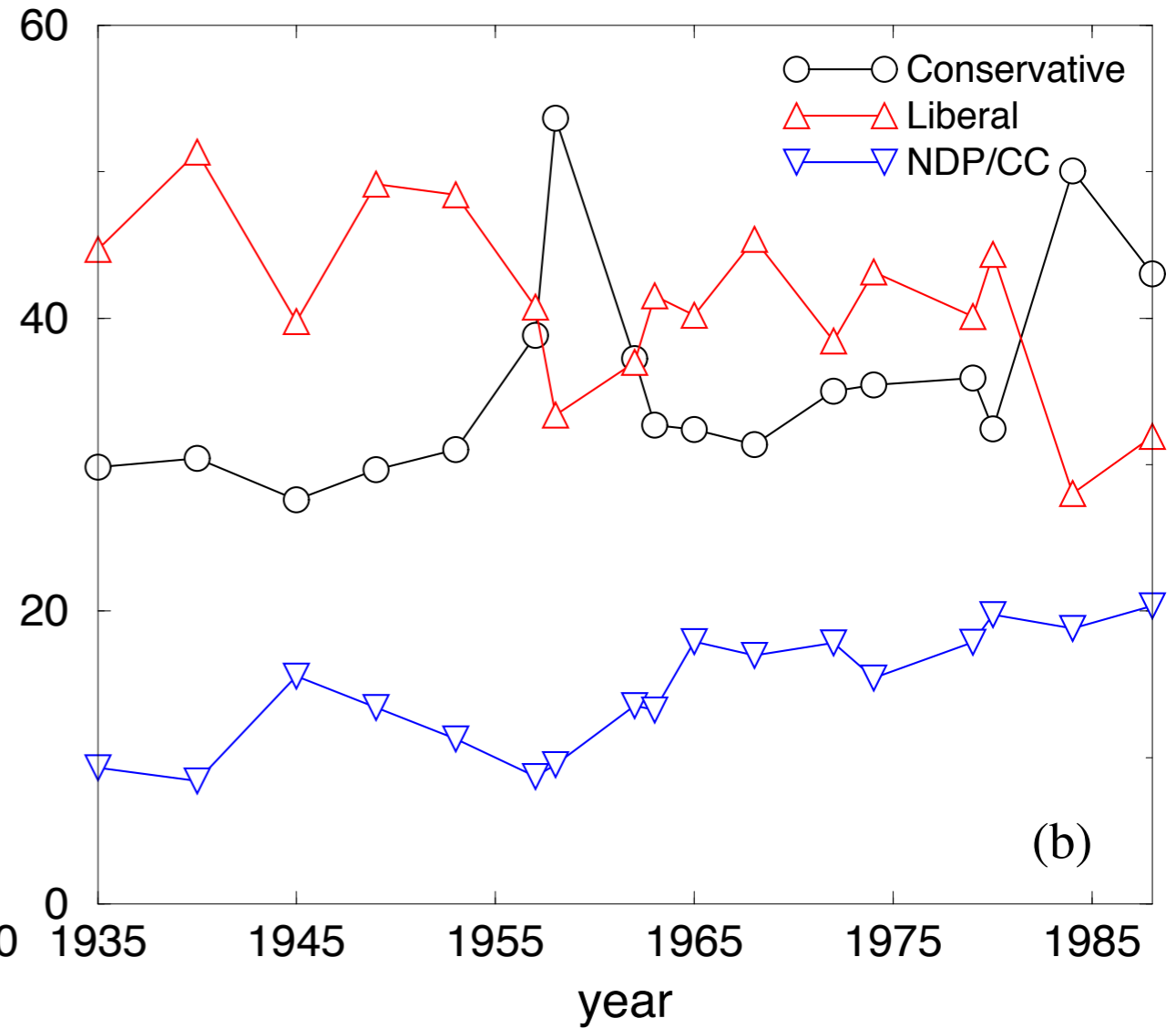
fast ($< N$)
consensus

Strategic Voting

British election results since 1830



Canadian election results since 1935



Strategic Voter Model

D.Volovik, M. Mobilia, SR
EPL **85**, 48001 (2009)

randomly-selected voter changes to any other state equiprobably (rate T)

majority-minority interaction: minority *preferentially* changes to majority (rate r)

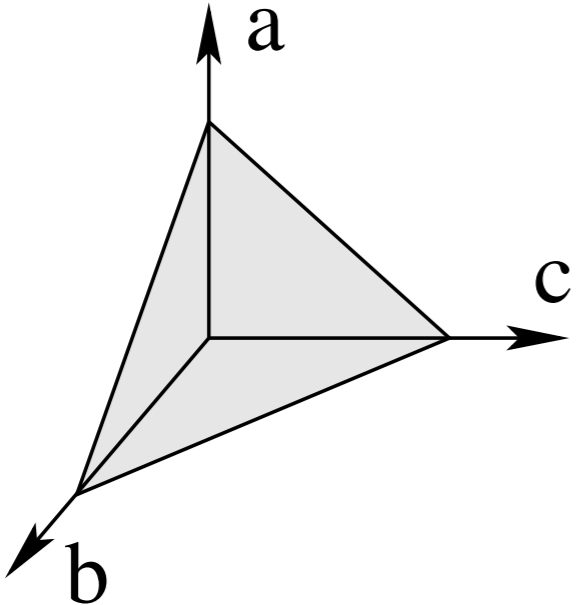
rate equations (A, B majority; c minority):

$$\dot{A} = T(B + c - 2A) + r Ac$$

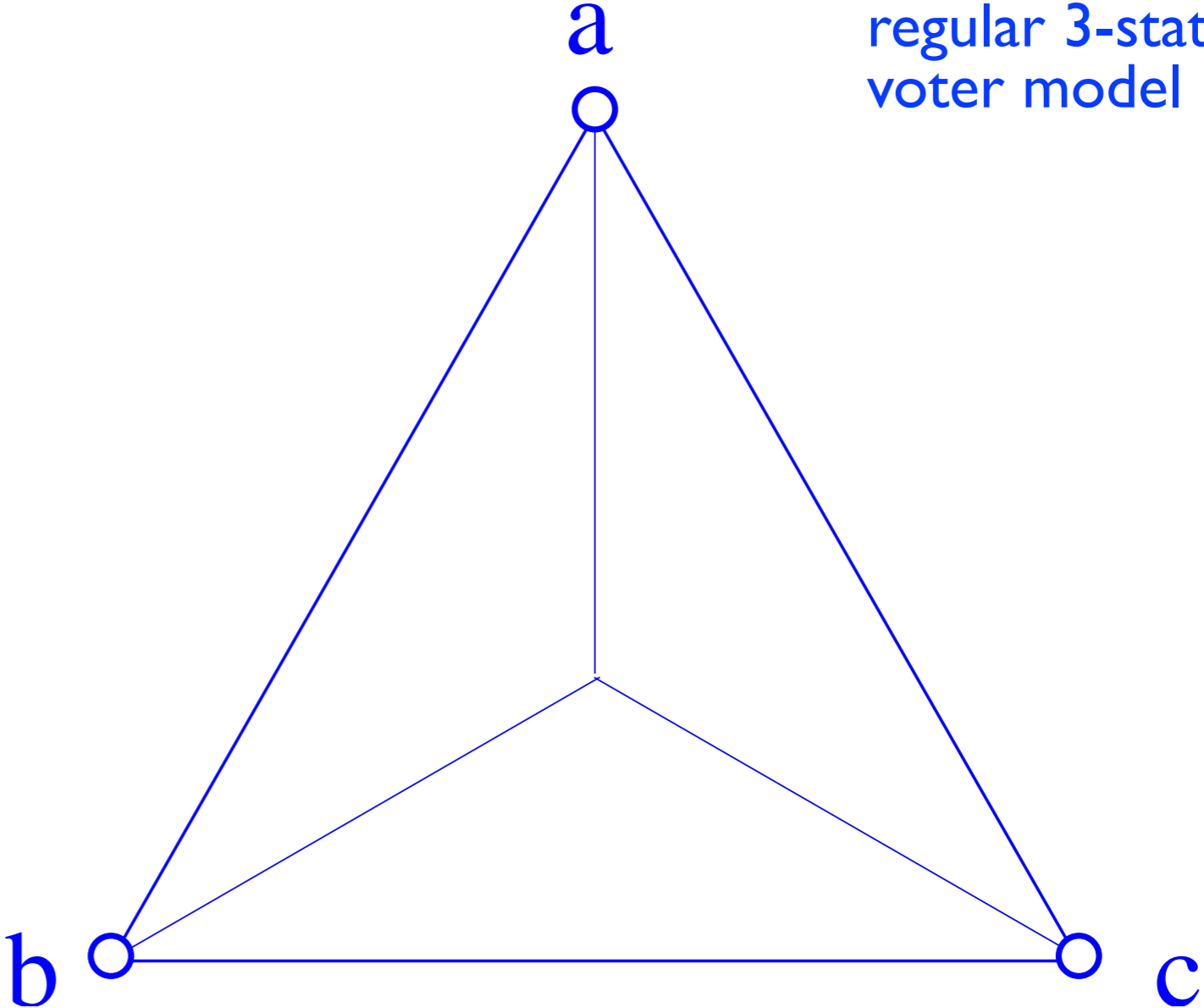
$$\dot{B} = T(c + A - 2B) + r Bc$$

$$\dot{c} = T(A + B - 2c) - r (A + B)c$$

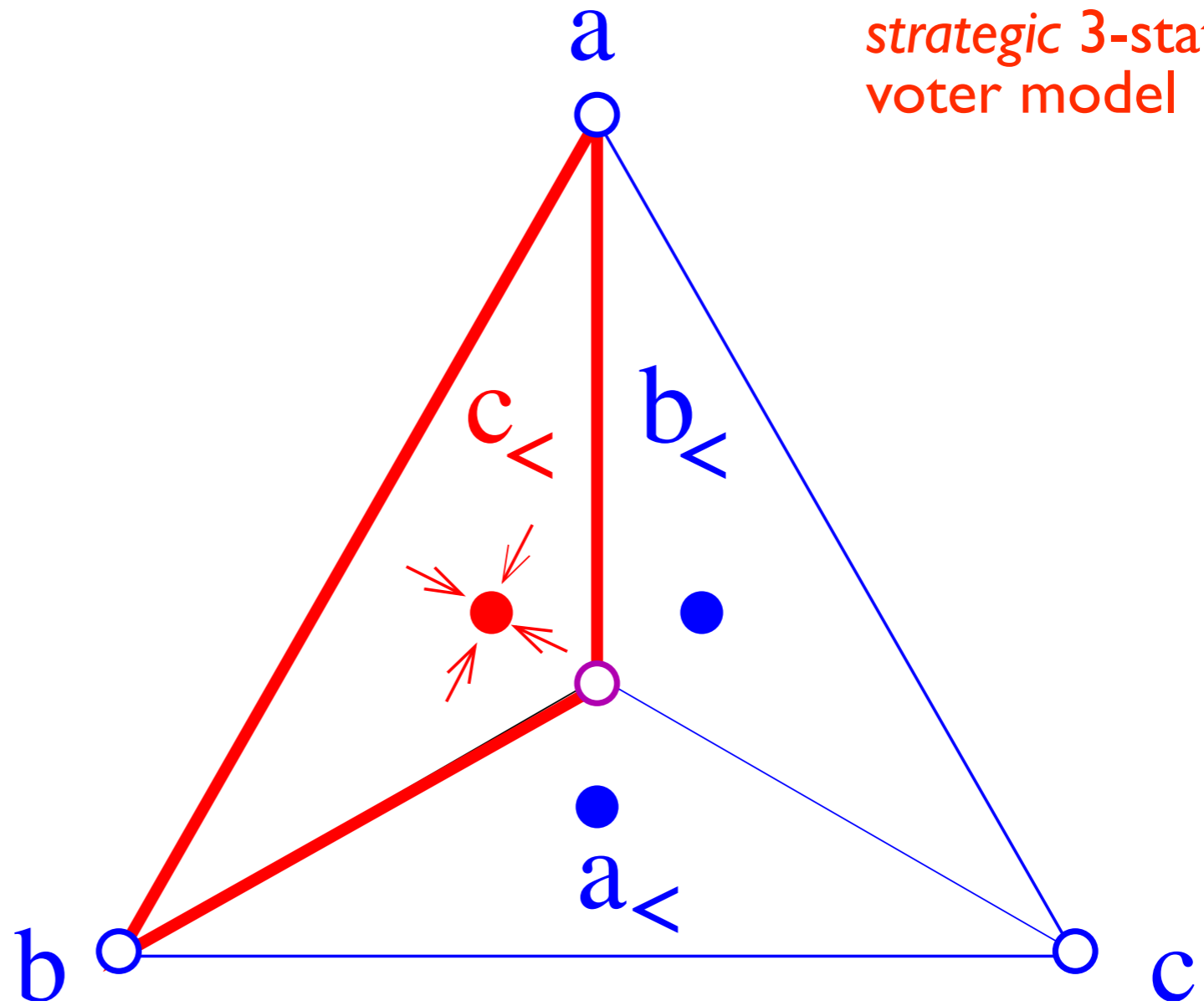
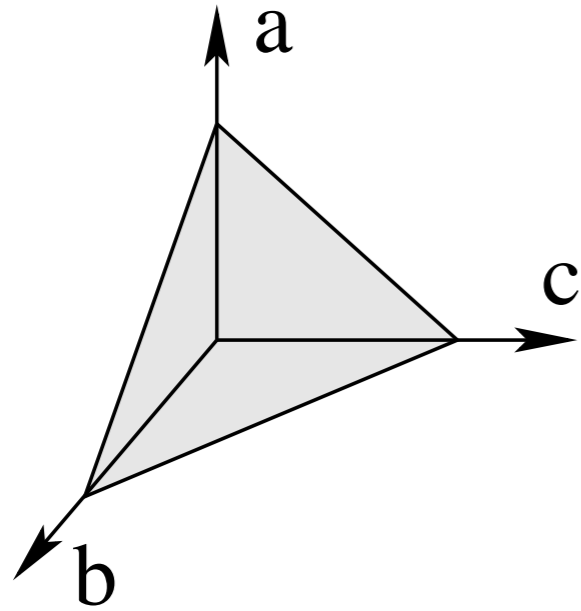
Phase Portrait



regular 3-state
voter model

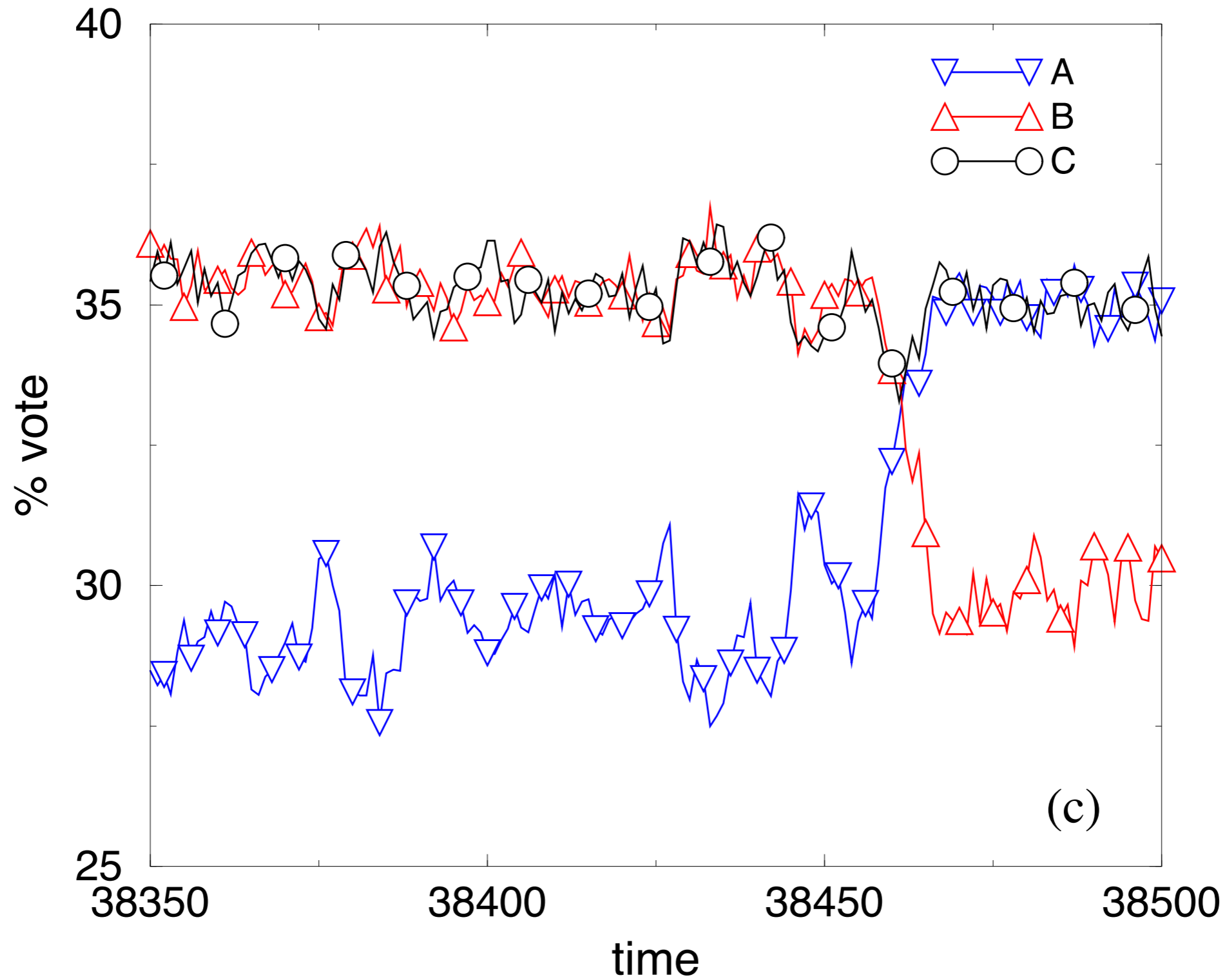


Phase Portrait

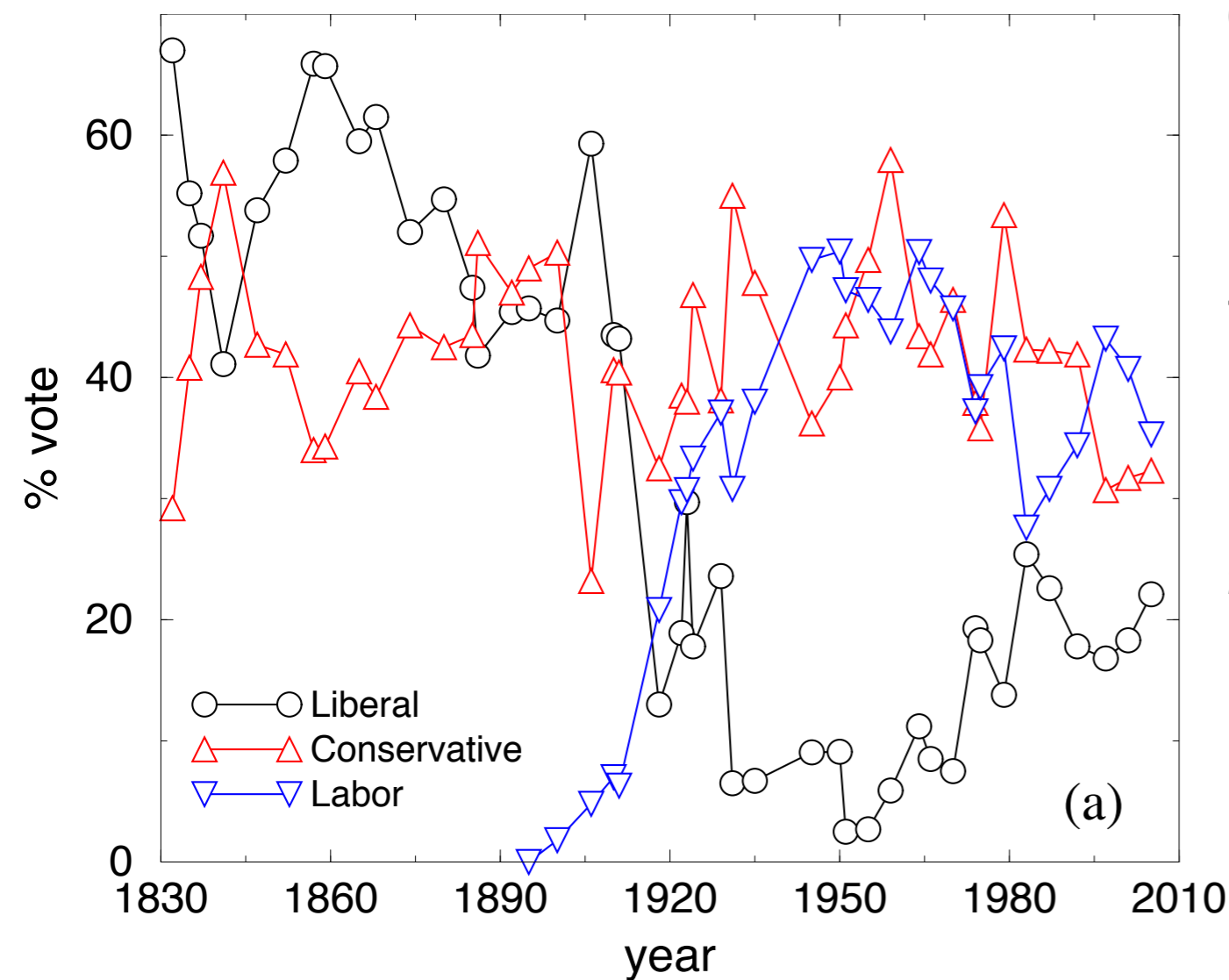


*strategic 3-state
voter model*

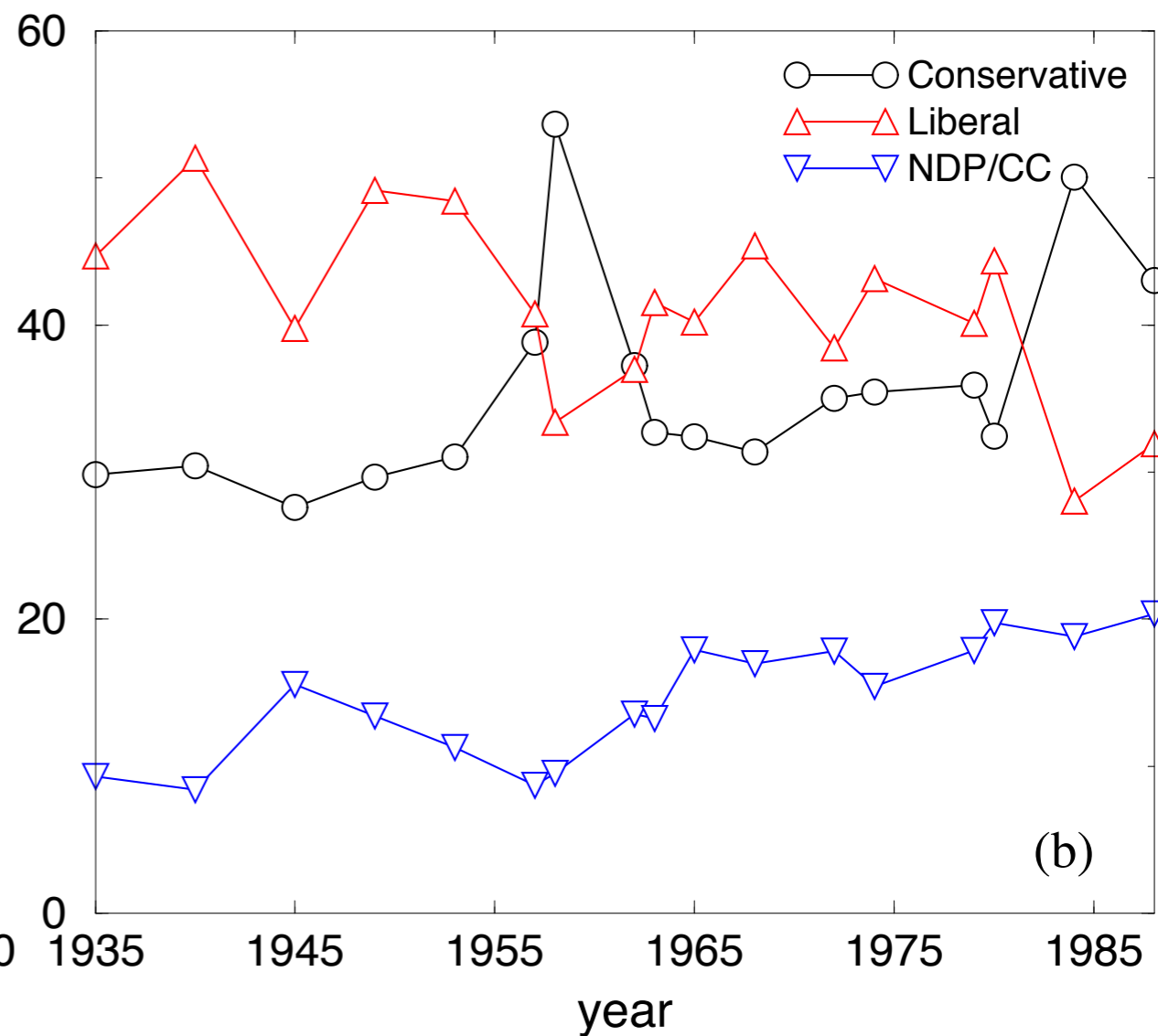
Slow Switching



British election results since 1830



Canadian election results since 1935



Partisan Voter Model

N. Masuda, N. Gibert, SR
arXiv:1003.0768

Partisan Voter Model

N. Masuda, N. Gibert, SR
arXiv:1003.0768

↑ happy
democrat
density D_h

↓ sad
democrat
density D_s

↑ sad
republican
density R_s

↓ happy
republican
density R_h

Partisan Voter Model

N. Masuda, N. Gibert, SR
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democrat
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republican
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density R_h

partisan voting update:

Partisan Voter Model

N. Masuda, N. Gibert, SR
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partisan voting update:

1. Pick voter, pick neighbor (as in usual voter model);

Partisan Voter Model

N. Masuda, N. Gibert, SR
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↑ happy democrat
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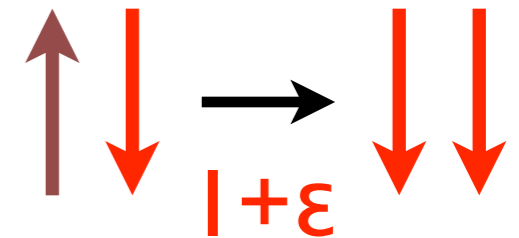
↑ sad republican
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partisan voting update:

1. Pick voter, pick neighbor (as in usual voter model);

2a. If initial voter becomes *happy* by adopting neighboring state, change occurs with rate $1+\varepsilon$;



Partisan Voter Model

N. Masuda, N. Gibert, SR
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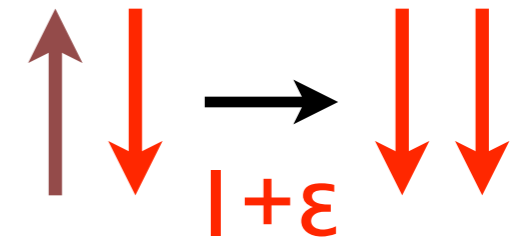
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density R_h

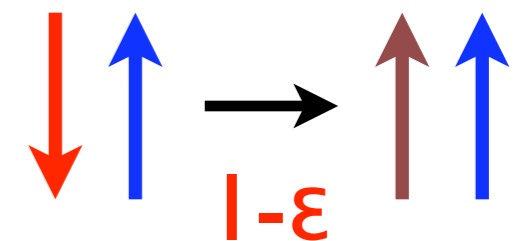
partisan voting update:

1. Pick voter, pick neighbor (as in usual voter model);

2a. If initial voter becomes *happy* by adopting neighboring state, change occurs with rate $1+\varepsilon$;



2b. If initial voter becomes *unhappy* by adopting neighboring state, change occurs with rate $1-\varepsilon$.



Partisan Voter Model: Mean-Field Limit

rate equations:

$$\dot{D}_h = 2\epsilon D_h D_s + (1 + \epsilon) D_s R_s - (1 - \epsilon) D_h R_h$$

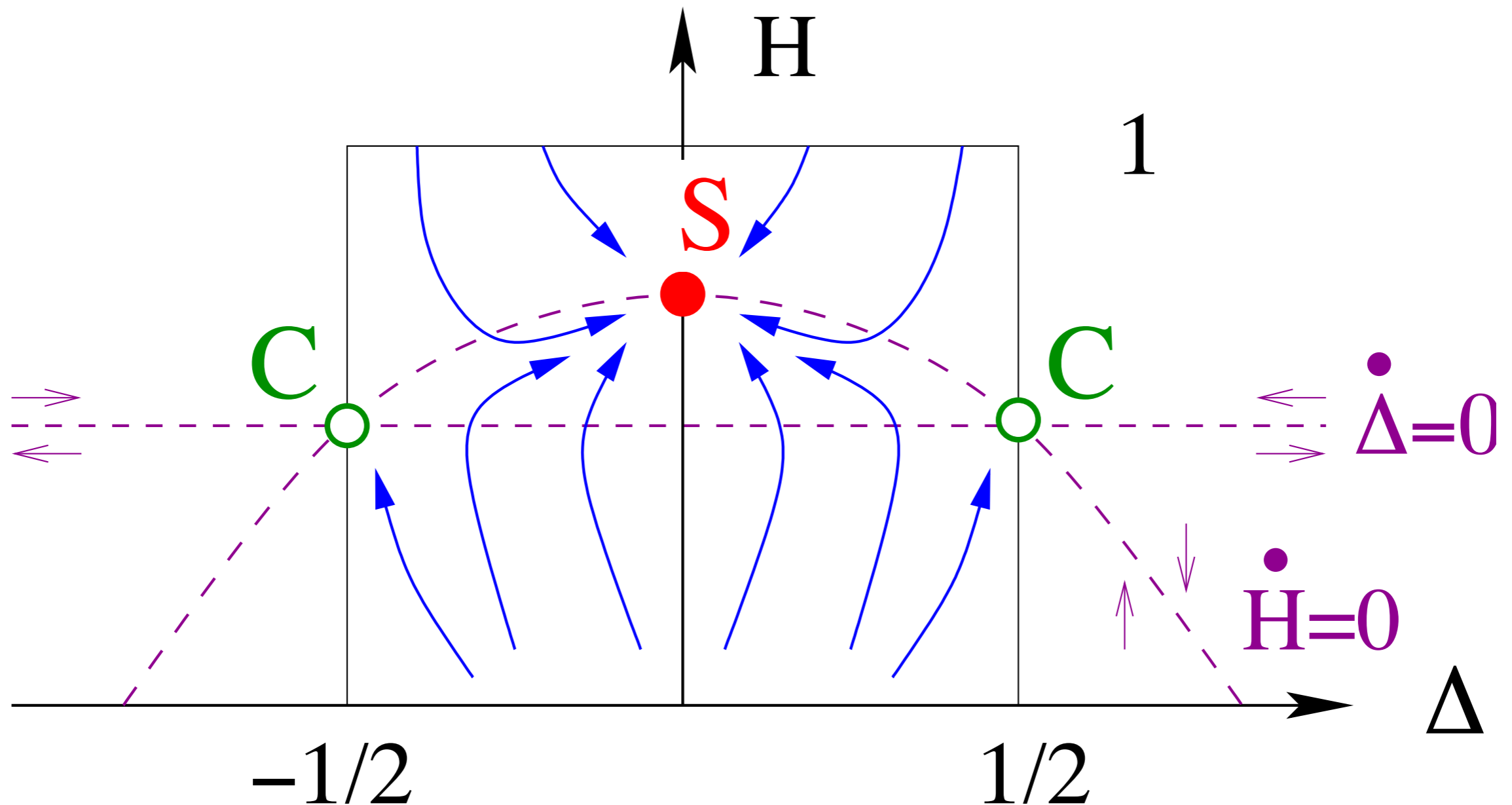
$$\dot{D}_s = -2\epsilon D_h D_s + (1 - \epsilon) D_h R_h - (1 + \epsilon) D_s R_s$$

and $R \leftrightarrow D$

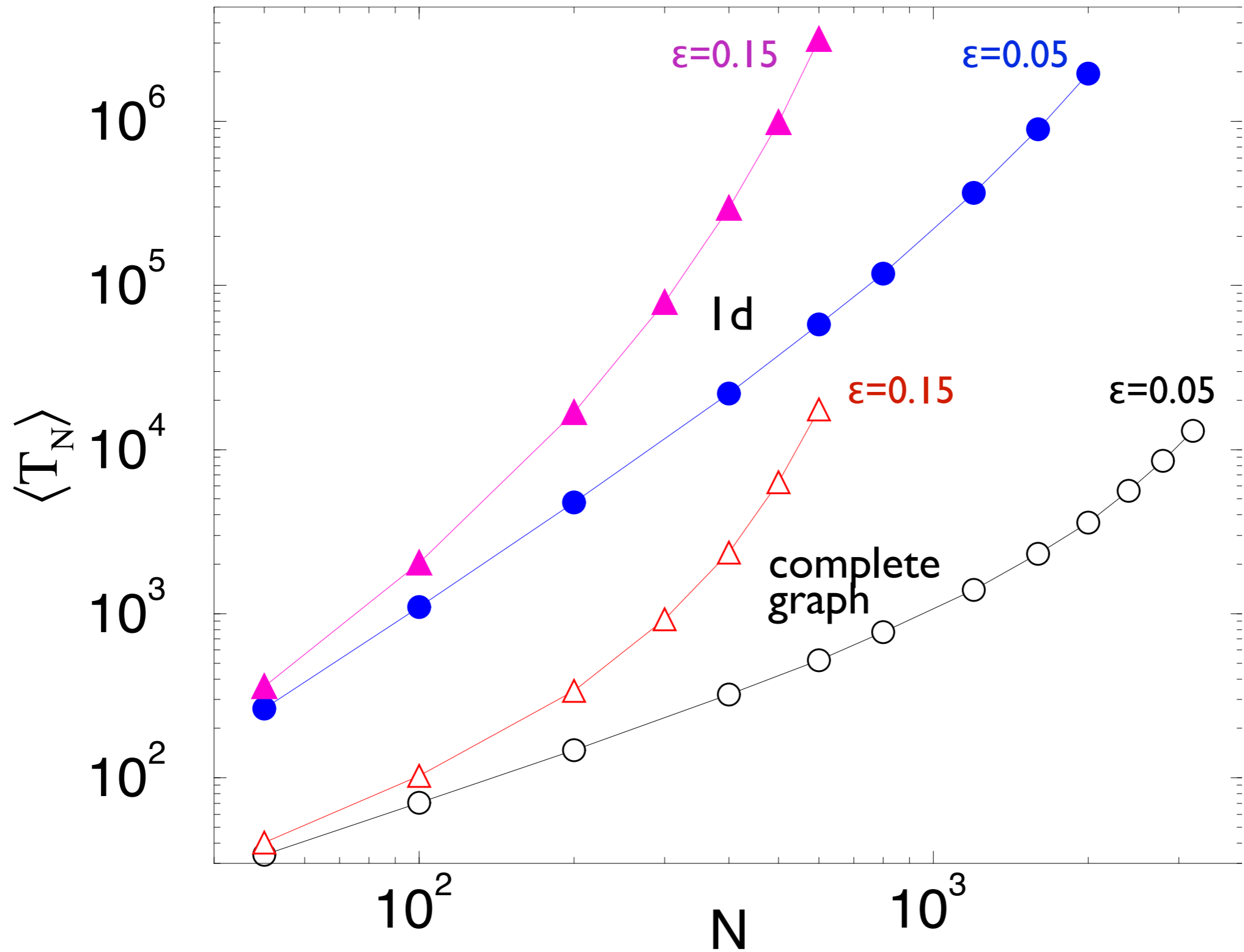
Symmetric Case: $D=R=1/2$

$H \equiv D_h + R_h$
 = density of happy voters

$\Delta \equiv D_h - R_h = D_h - (\frac{1}{2} - R_s) = \rho - \frac{1}{2}$
 = density democratic voters $-\frac{1}{2}$



Consensus Time on Finite Graphs



Summary & Outlook

Voter model:

paradigmatic, soluble, (but hopelessly naive)

Voter model on complex networks:

new conservation law

meandering route to consensus

fast consensus for broad degree distributions

Extensions:

strategic voting → minority suppressed

partisan voting → selfishness forestalls consensus

Future:

“churn” rather than consensus

heterogeneity of real people

positive and negative social interactions → social balance

Crass Commercialism

Aimed at graduate students, this book explores some of the core phenomena in non-equilibrium statistical physics. It focuses on the development and application of theoretical methods to help students develop their problem-solving skills.

The book begins with microscopic transport processes: diffusion, collision-driven phenomena, and exclusion. It then presents the kinetics of aggregation, fragmentation and adsorption, where the basic phenomenology and solution techniques are emphasized. The following chapters cover kinetic spin systems, both from a discrete and a continuum perspective; the role of disorder in non-equilibrium processes; hysteresis from the non-equilibrium perspective; the kinetics of chemical reactions; and the properties of complex networks. The book contains 200 exercises to test students' understanding of the subject. A link to a website hosted by the authors, containing supplementary material including solutions to some of the exercises, can be found at www.cambridge.org/9780521851039.

Pavel L. Krapivsky is Research Associate Professor of Physics at Boston University. His current research interests are in strongly interacting many-particle systems and their applications to kinetic spin systems, networks, and biological phenomena.

Sidney Redner is a Professor of Physics at Boston University. His current research interests are in non-equilibrium statistical physics, and its applications to reactions, networks, social systems, biological phenomena, and first-passage processes.

Eli Ben-Naim is a member of the Theoretical Division and an affiliate of the Center for Nonlinear Studies at Los Alamos National Laboratory. He conducts research in statistical, nonlinear, and soft condensed-matter physics, including the collective dynamics of interacting particle and granular systems.

Cover illustration: Snapshot of a collision cascade in a perfectly elastic hard-sphere gas in two dimensions due to a single incident particle. Shown are the cloud of moving particles (red) and the stationary particles (blue) that have not yet experienced any collisions. Figure courtesy of Tibor Antal.

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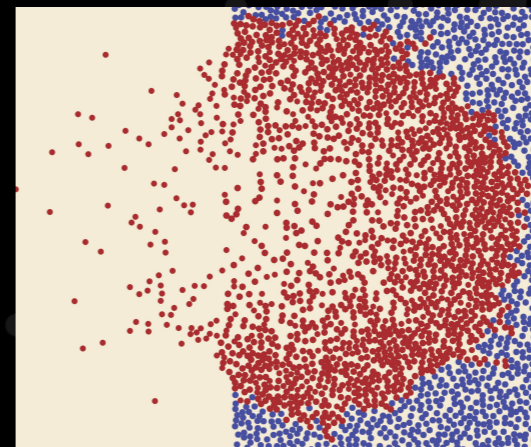


Krapivsky,
Redner and
Ben-Naim

A Kinetic View of STATISTICAL PHYSICS

CAMBRIDGE

A Kinetic View of STATISTICAL PHYSICS



Paul L. Krapivsky, Sid Redner
and Eli Ben-Naim

*to appear
this October*

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