Coarsening, Slow Dynamics, & Freezing in the Simplest Spin Systems

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Basic question: What is the final state of the Ising-Glauber model @ T=0 with symmetric initial conditions?

We expect: Ground state is approached as $t \to \infty$

Basic results: 1.

dimension	expectation
1	correct
2	correct "sort of"
>2	wrong

2. Multiscale relaxation, freezing, & related strange features

New direction: *Majority model* Ground state is always reached, but with strange dynamics

The System

Ising Hamiltonian $\mathcal{H} = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$

Initial state:

$$\uparrow \qquad \text{with probability } p$$

$$\downarrow \qquad \text{with probability } 1 - p$$

Lattice: even co-ordination number, periodic boundaries

Dynamics: Glauber at T=0: Pick a random spin and consider outcome of a reversal

$$\text{if} \quad \Delta E < 0 \qquad \text{do it}$$

$$\text{if} \quad \Delta E > 0 \qquad \text{don't do it}$$

$$\text{if} \quad \Delta E = 0 \qquad \text{do it with prob. } 1/2$$

Results in d=1

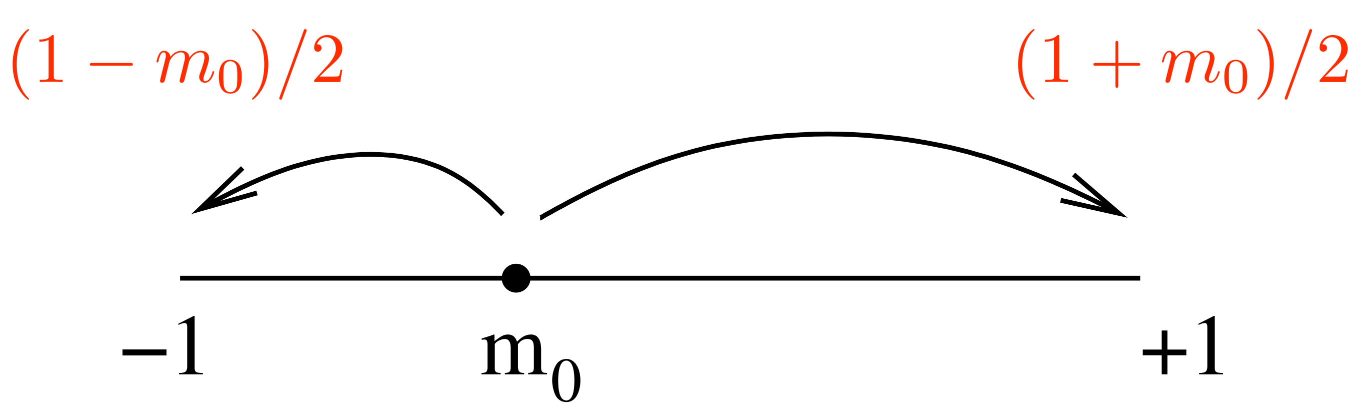
Equation of motion at T=0 (with Glauber kinetics):

$$\dot{s}_j = -s_j + \frac{1}{2}(s_{j-1} + s_{j+1}), \qquad \text{where } s_j = \langle \sigma_j \rangle$$

Hence $\quad \langle \dot{m} \rangle = \sum_j \dot{s}_j = 0 \quad \rightarrow \quad \langle m \rangle$ conserved

m diffuses

Pictorial representation:



ultimate hitting probabilities

Summary of results in d=2

(Spirin, Krapvisky, & SR 01)

Final state:

$$\begin{cases} \text{ground state} & \text{prob.} \approx 2/3 \\ \text{stripe} & \text{prob.} \approx 1/3 \end{cases}$$

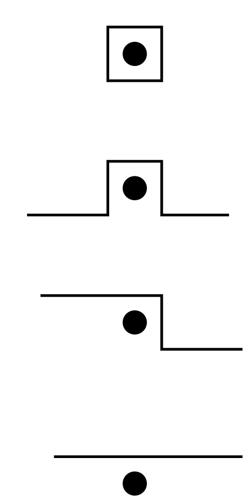
Survival probability: 2 time scales!

$$\begin{aligned} M_k &\equiv \langle t^k \rangle^{1/k} \\ &\sim \begin{cases} L^{3.5} & k > 1 \\ L^{2^+} & k < 1 \end{cases} \end{aligned}$$

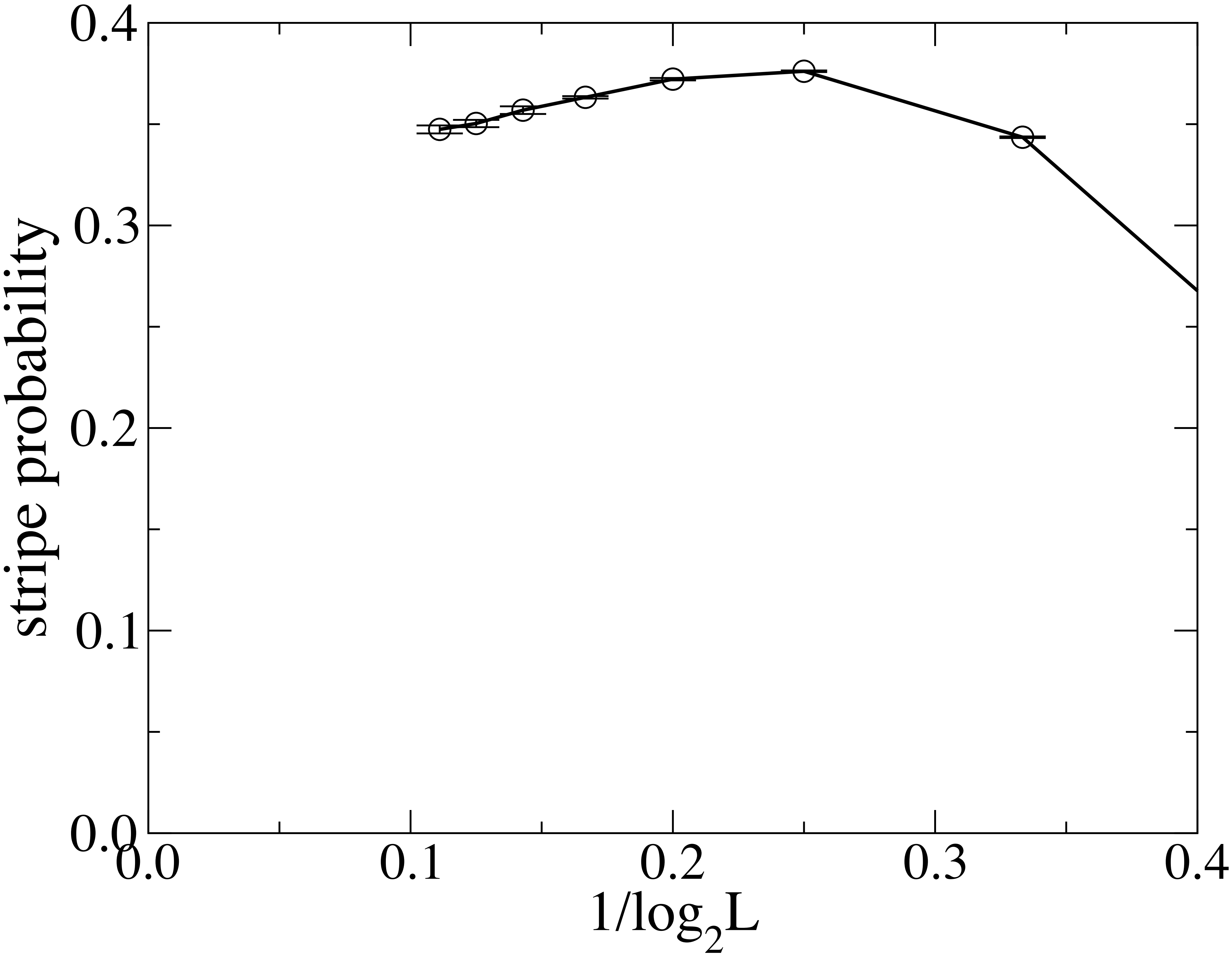
Energy evolution:

$$\begin{aligned} E(t) &\sim t^{-1/2} \\ n_E(t) &\sim t^{-\mu(E)} \end{aligned}$$

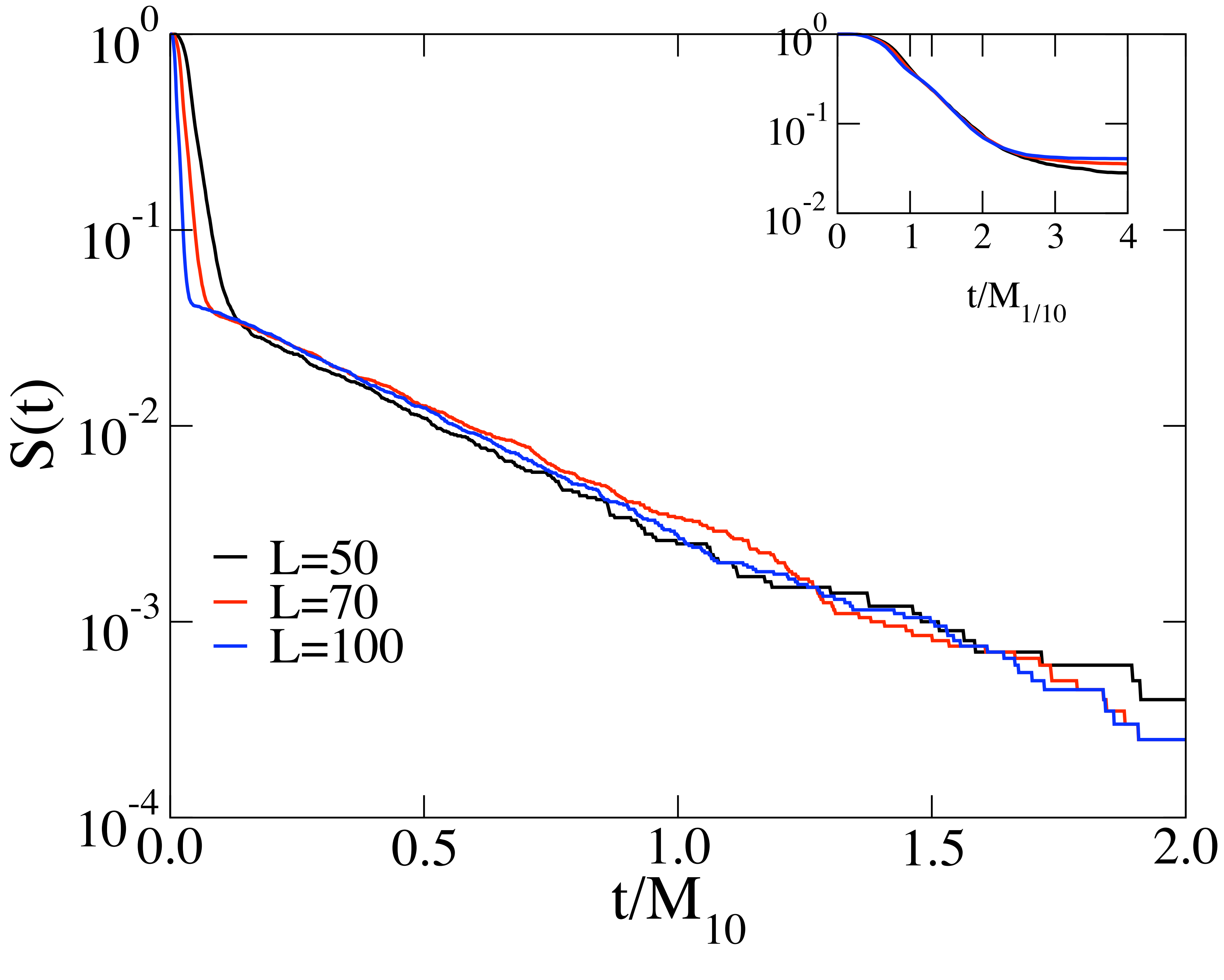
$$\begin{aligned} \mu(+4) &\approx 2.1 \\ \mu(+2) &\approx 1.4 \\ \mu(0) &\approx 0.5 \\ \mu(-2) &\approx 0.45 \end{aligned}$$



The final state in 2d



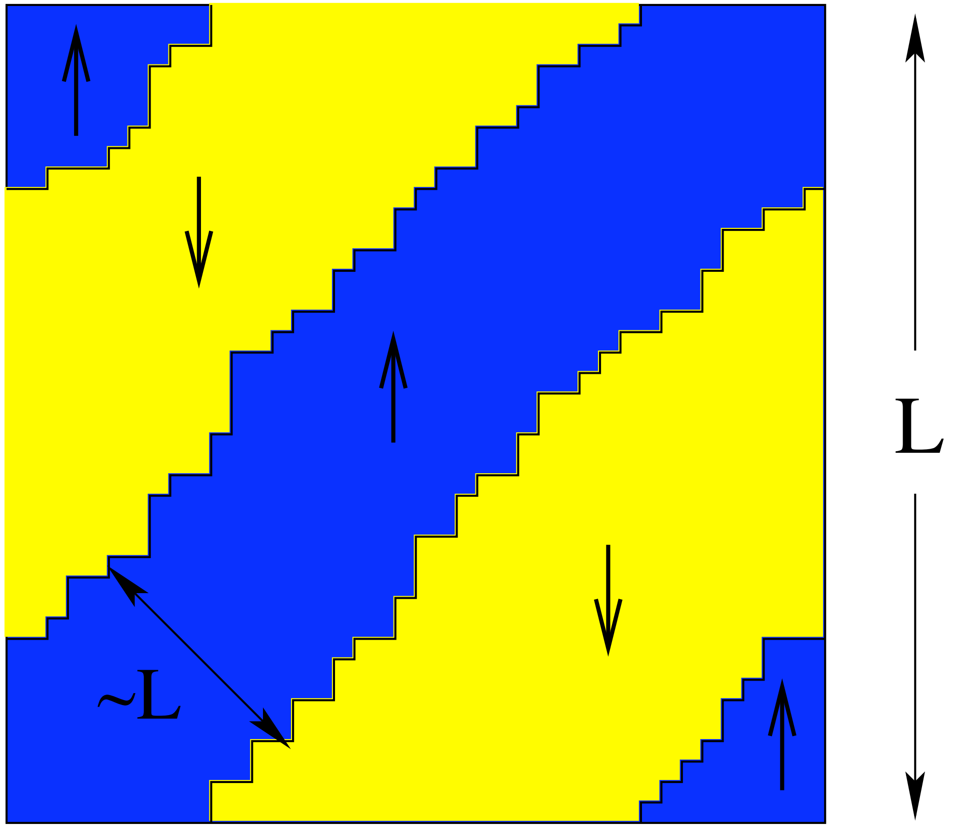
Survival probability



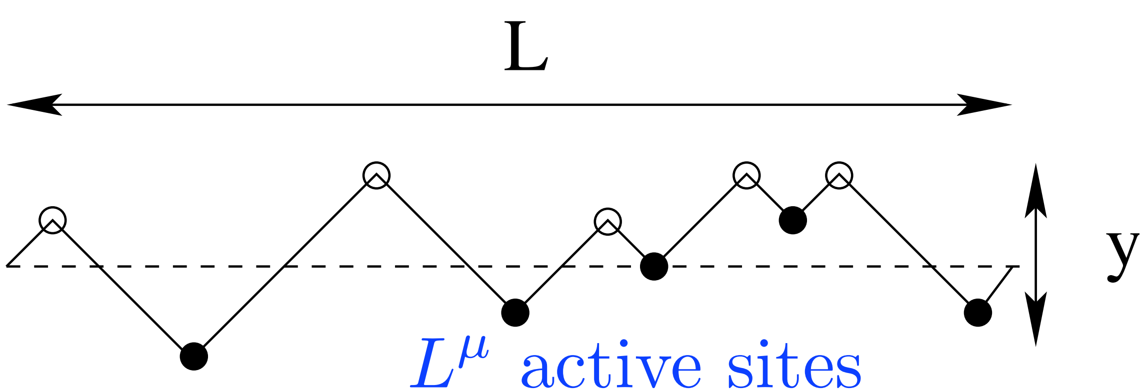
Understanding the two time-scale relaxation

We observe: 95% short-lived, 5% long lived!

Why? Diagonal stripe!



Diagonal stripe dynamics: (Plischke et al 87)



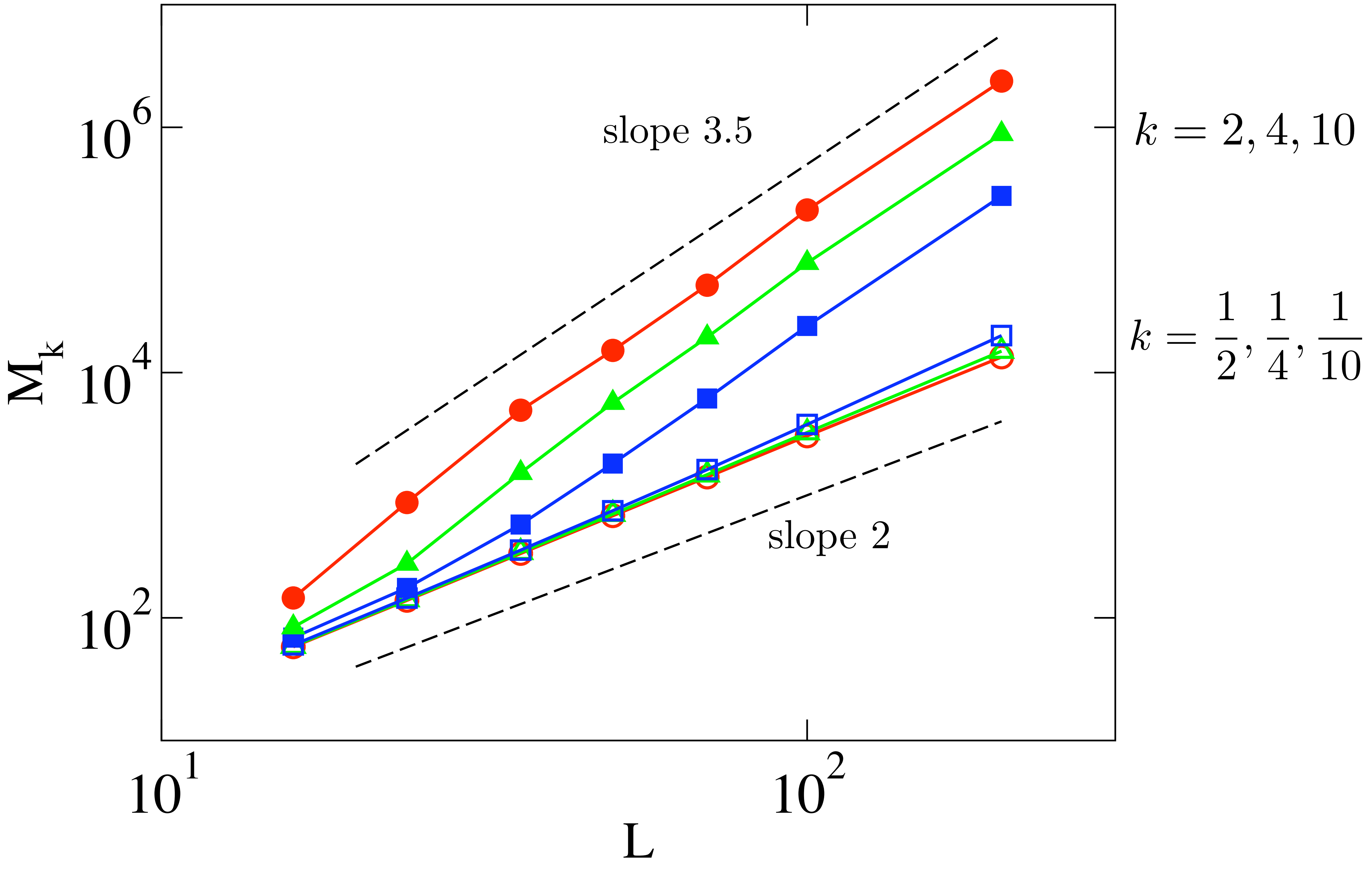
$$\Delta t = 1, \quad L^{\mu} \text{ events} \quad \rightarrow \quad \Delta y_{\text{cm}} \sim L^{\mu/2}/L$$

$$\rightarrow \quad D(L) \sim L^{\mu-2}$$

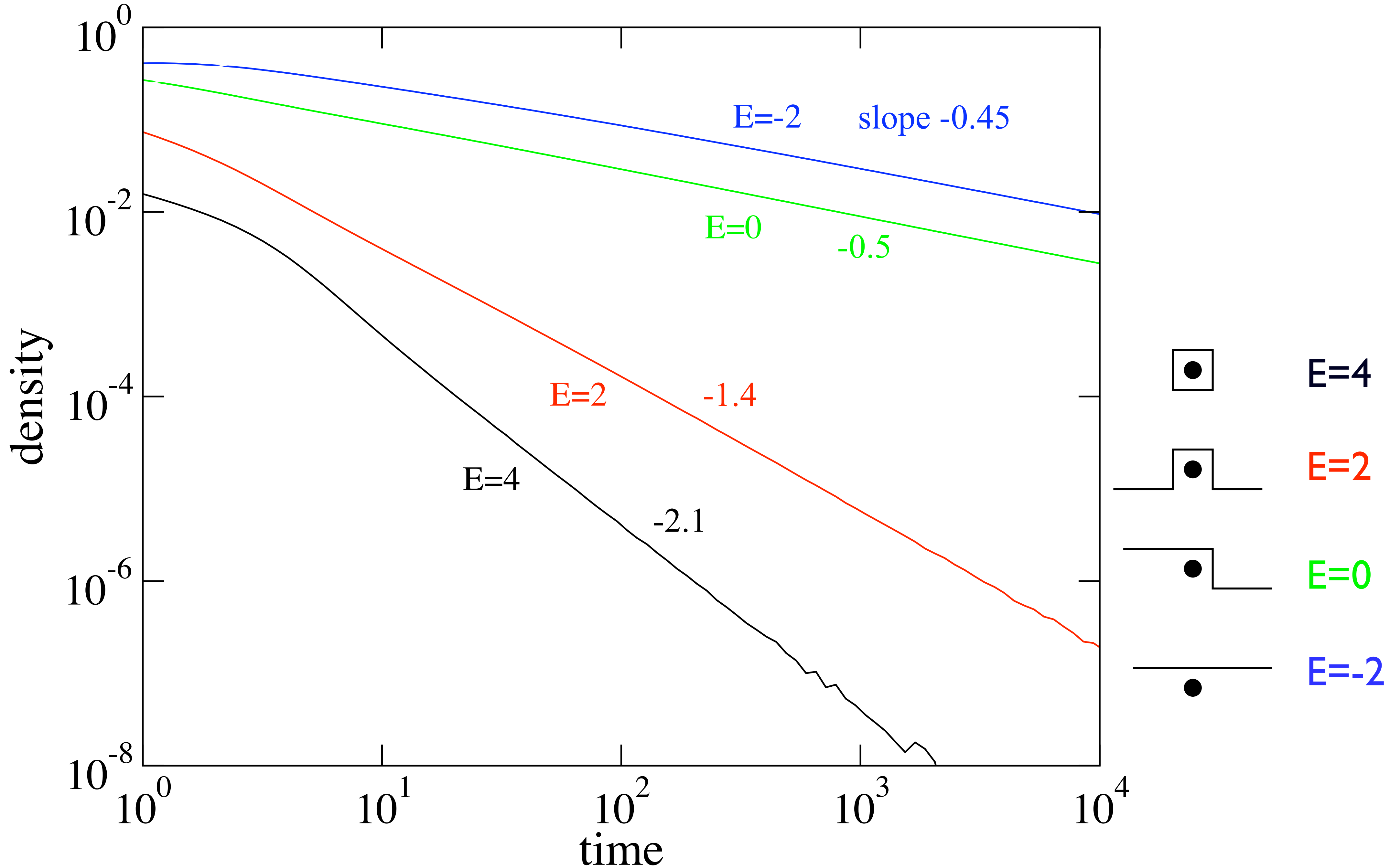
$$\text{survival time } \tau \sim L^2/D \quad \sim \quad L^{4-\mu} \quad \text{but } \mu = 1/2$$

$$\sim \quad L^{3.5}$$

Multiscaling in moments of the stopping time

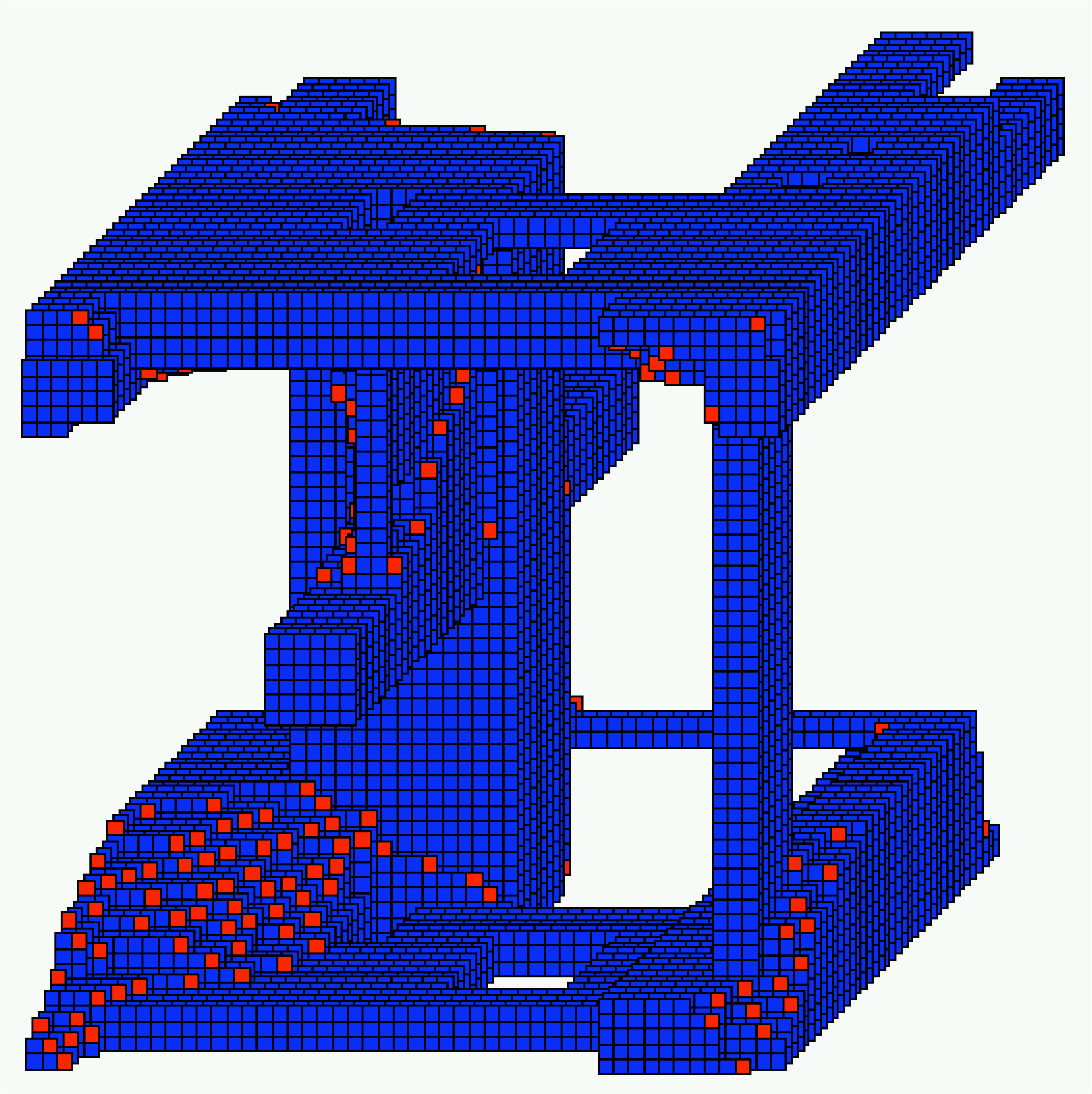


Densities of fixed-energy spins



Higher dimensions: Ground state never reached!

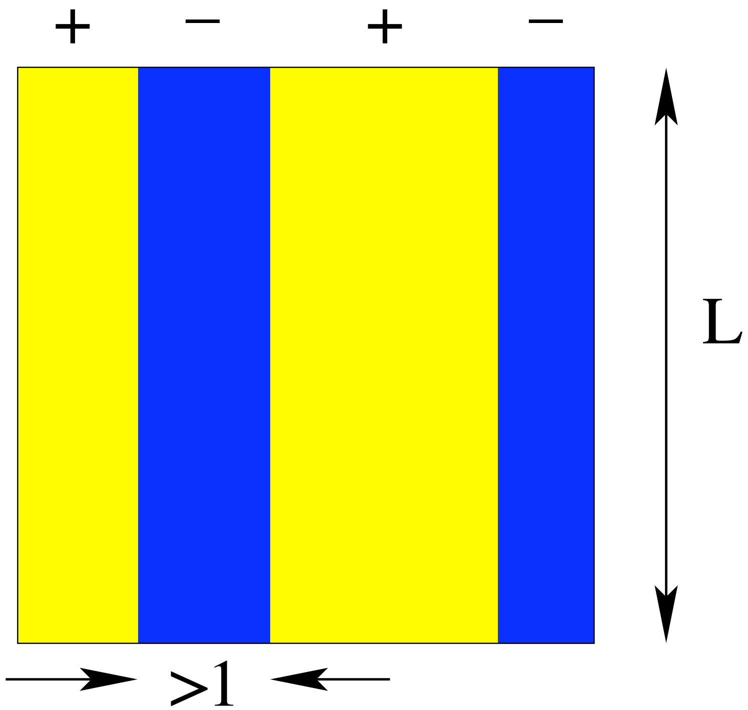
Final “sponge” state in 3d:



Why does the system get stuck?

Proliferation of metastable states as d increases.

d=2: stripe packing

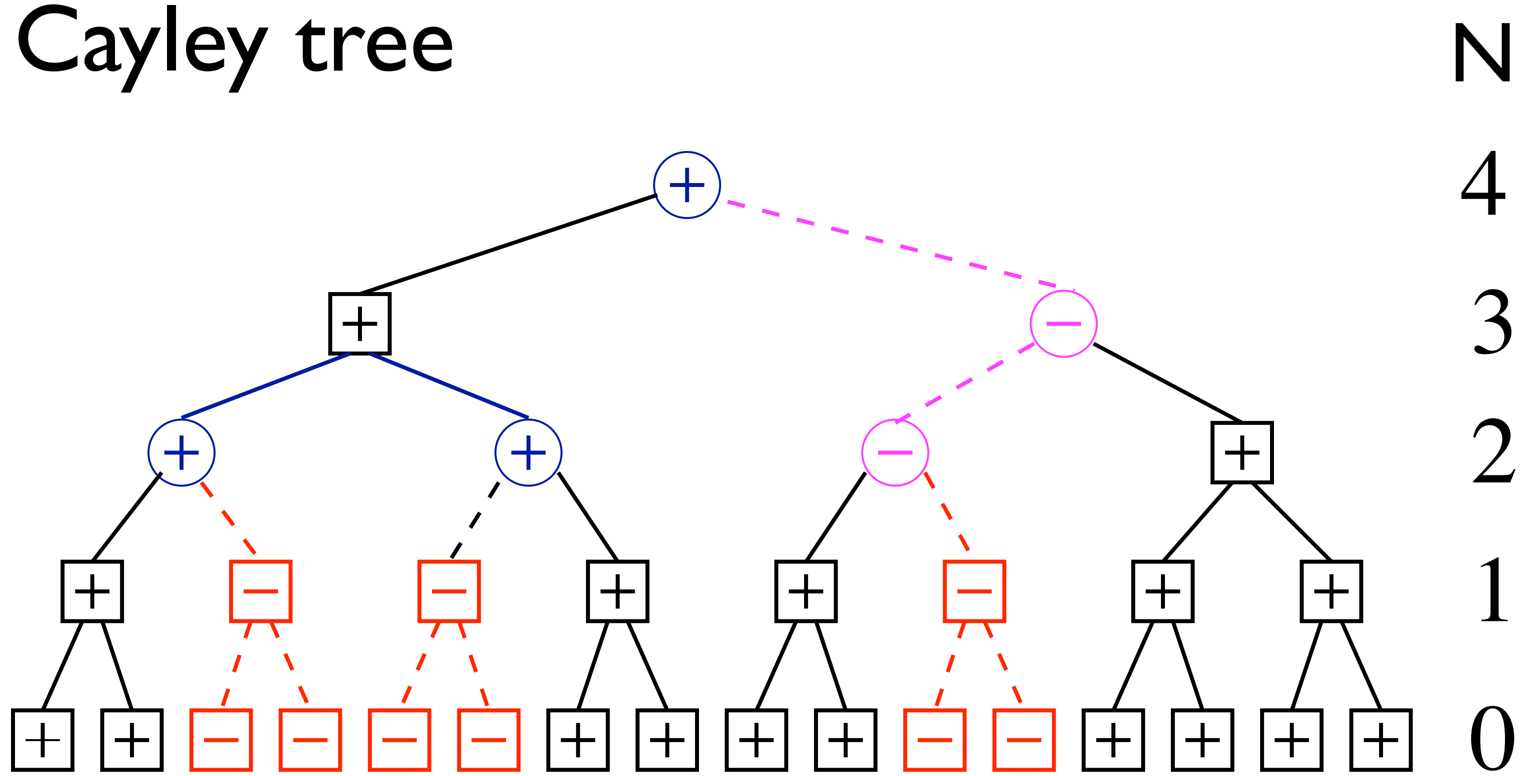


$$M_L \sim e^{aL} \sim e^{aV^{1/2}}$$

d=3: filament packing

$$M_L \sim e^{bL^2} \sim e^{bV^{2/3}}$$

d= ∞: Cayley tree



Recursion formulae for degeneracy

○ $$U_{N+1} = 2D_N \times \frac{1}{2} D_N = D_N^2$$ undetermined

□ $$D_{N+1} = \frac{1}{2} D_N^2 + \frac{1}{2} U_N^2 + 2 D_N U_N$$ determined

$$\rightarrow \ln M_{\mathcal{N}} \sim \ln(U_N + D_N) = \text{const.} \times \mathcal{N}$$

Cultural Interlude:
Two Fundamental Spin Models

Ising model with Glauber dynamics:

majority rule operating on a single spin: *system freezes!*
(above d=2)

Voter model:

1. Pick a random spin
2. Assume state of randomly-selected neighbor
3. Repeat until consensus

Exactly soluble in all dimension

dimension	consensus time
1	N^2
2	$N \ln N$
>2	N

d=2 is the critical dimension

Majority Rule

1. Pick a group of G spins (odd).
2. **All** spins in G adopt the majority state.
3. Repeat until consensus.

Eventual consensus always!

Basic questions:

1. *What is the final state?*
2. *What is the time until consensus?*

Mean-field solution (for G=3)

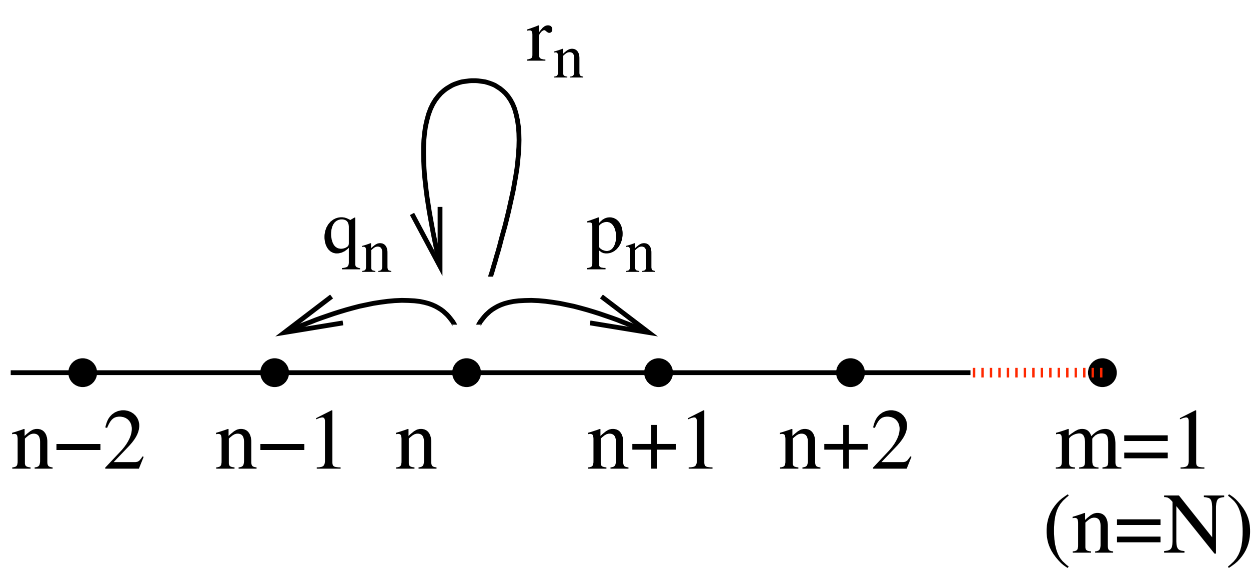
$$E_n \equiv \text{exit probability to } m = 1 \text{ starting from } n \text{ plus spins}$$

$$= p_n E_{n+1} + q_n E_{n-1} + r_n E_n$$

where $p_n = \binom{3}{2}\binom{N-3}{n-2}/\binom{N}{n}$

$$q_n = \binom{3}{1}\binom{N-3}{n-1}/\binom{N}{n}$$

$$r_n = 1 - p_n - q_n$$

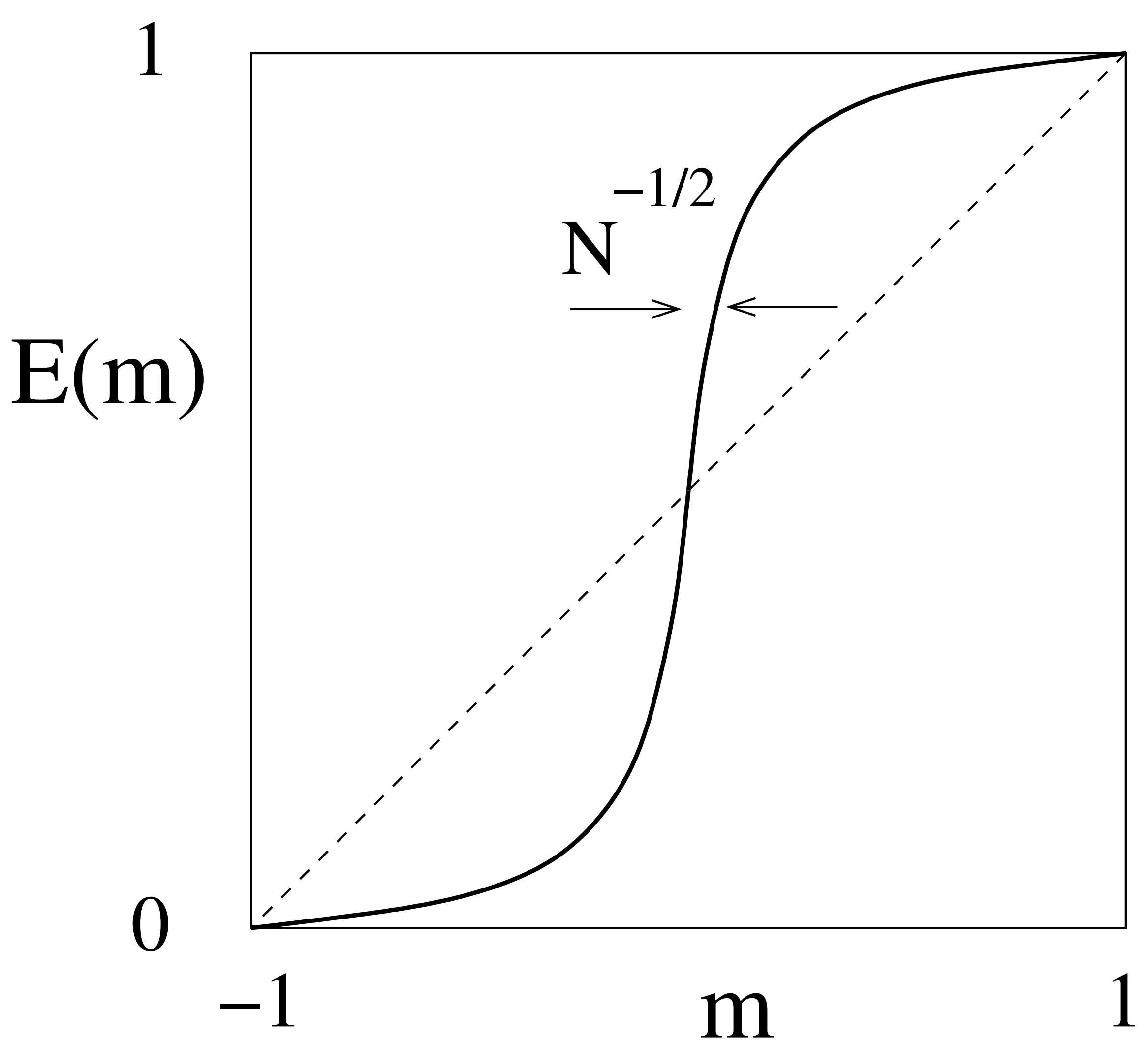


$$T_n \equiv \text{mean time to } m = 1 \text{ starting from } n \text{ plus spins}$$

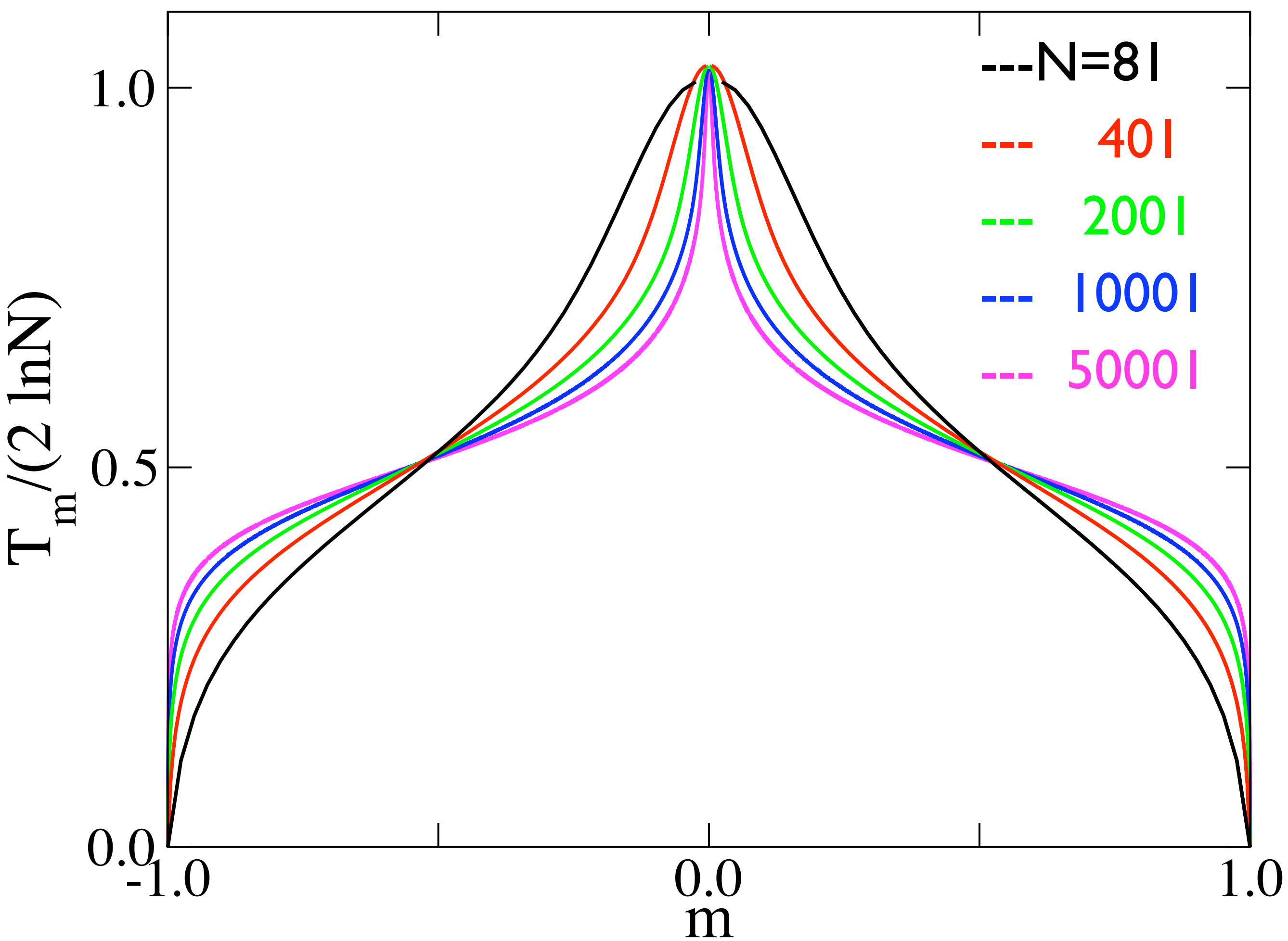
$$= p_n(T_{n+1} + \delta t) + q_n(T_{n-1} + \delta t) + r_n(T_n + \delta t)$$

Basic results for the mean-field limit

Exit probability
(schematic)

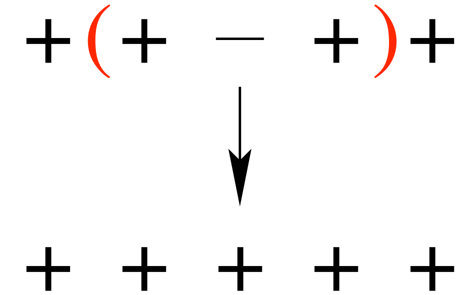


Consensus time
(data)



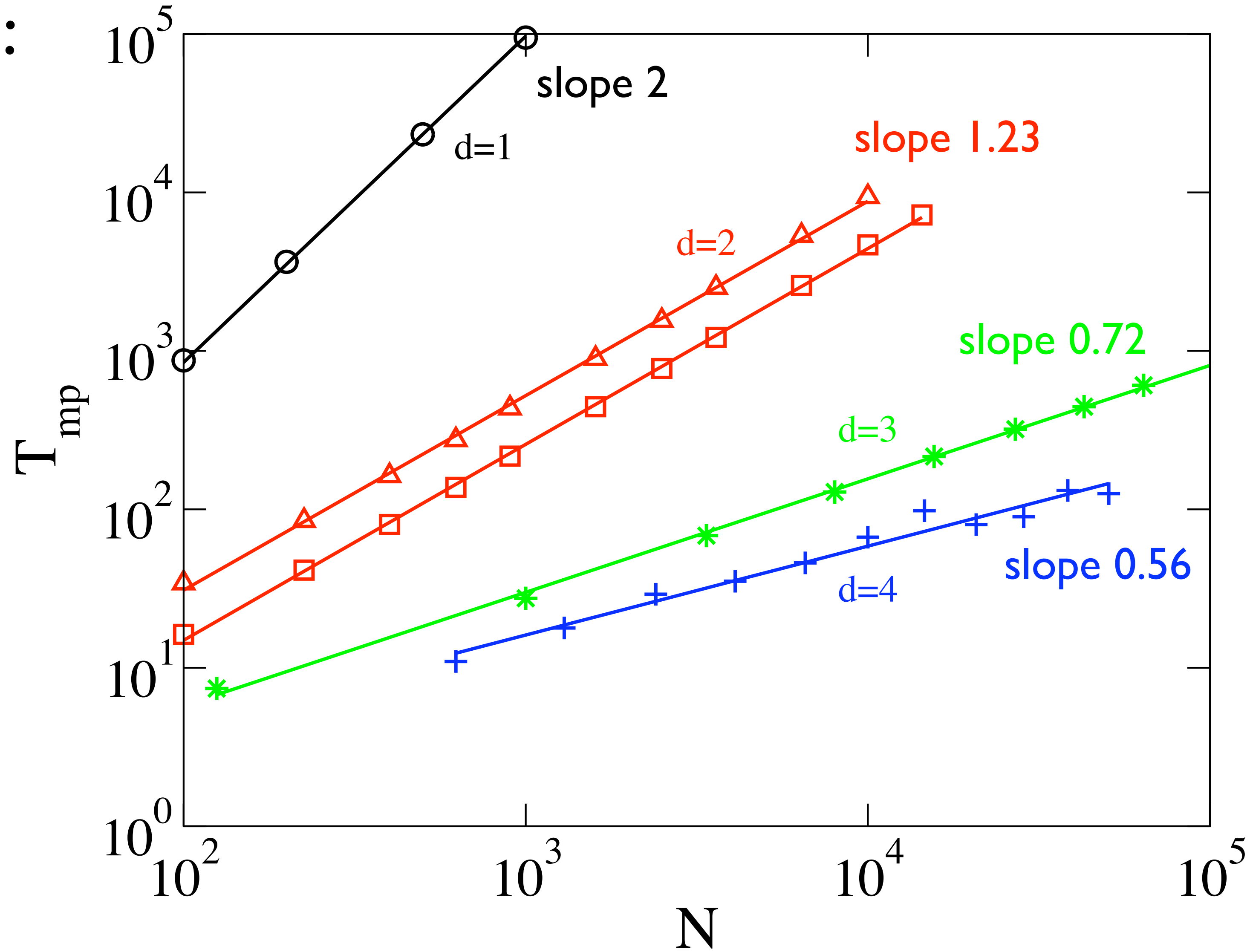
Finite spatial dimensions

d=1: *almost domain wall diffusion, but...*



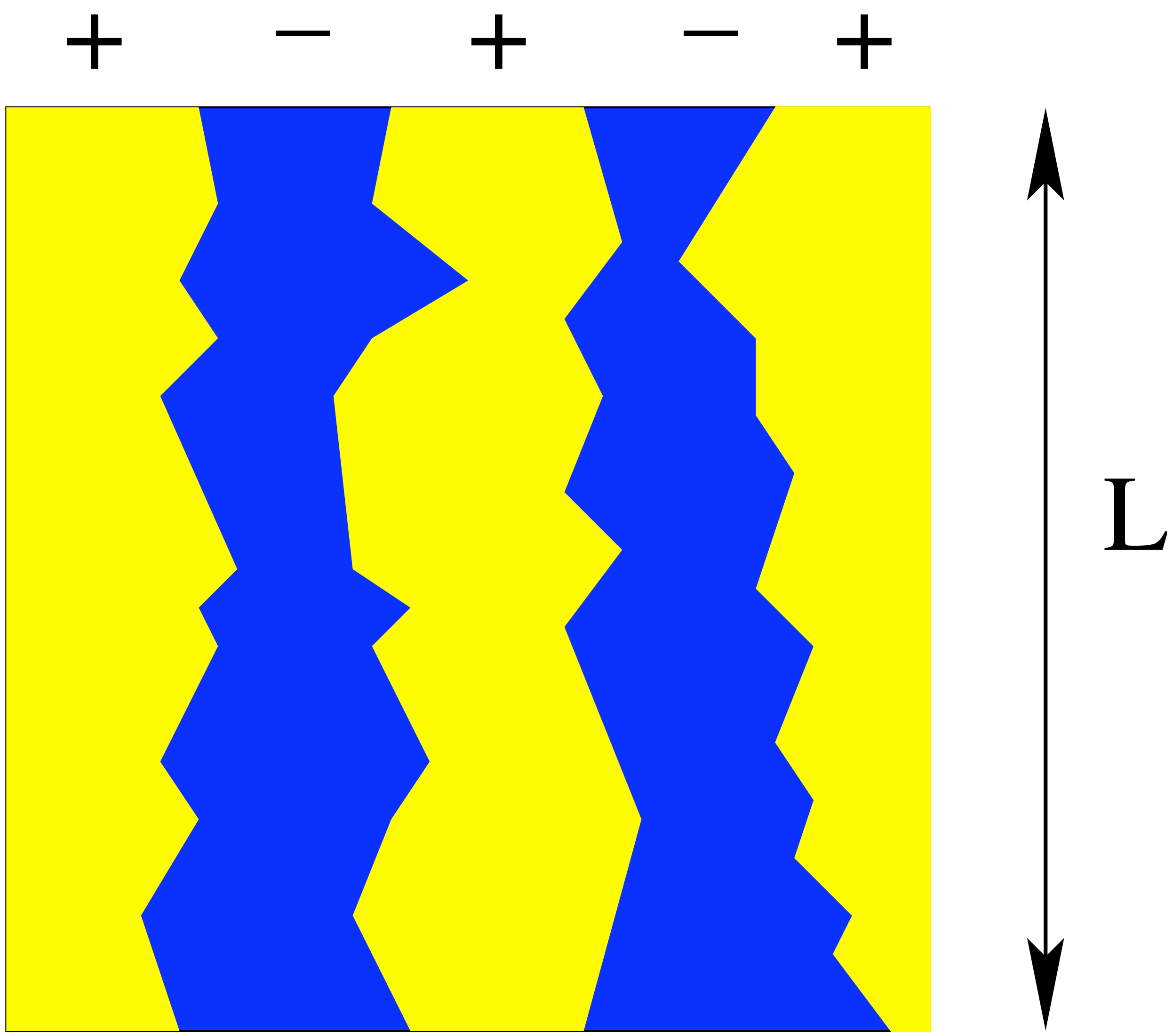
s disappear!

general d:



Critical dimension appears to be >4!

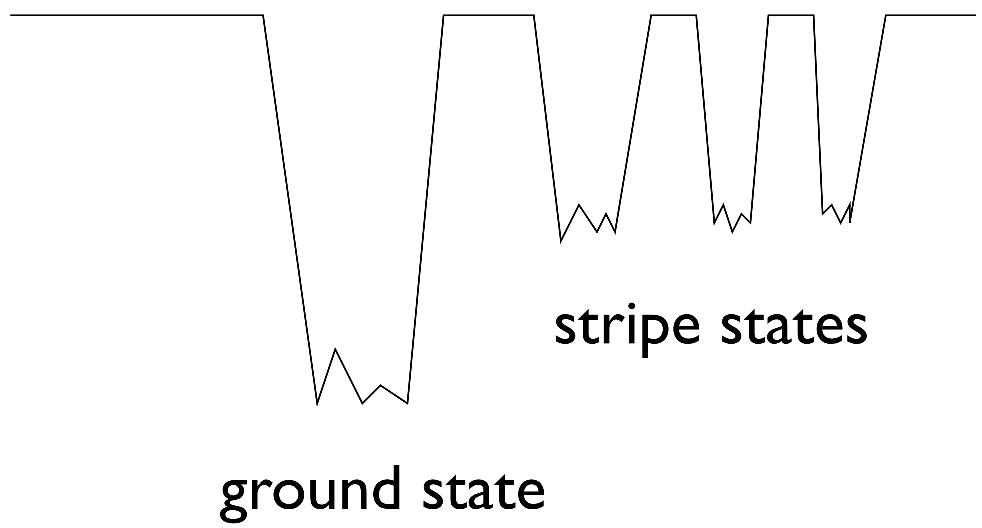
....and stripes still arise in 2d! ~50% of the time!



leads again to multiscale relaxation

Conclusions

1. Even the simplest Ising model has a complex evolution landscape



cartoon for d=2

2. How to characterize & quantify frozen states in d>2?

3. What happens for non-symmetric initial conditions?

4. New model: Majority rule

critical dimension appears to be >4 (perhaps infinite?)

complex relaxation in spite of a unique final state