# Winning and Losing in Competitions and Tournaments

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**Questions:** What types of class structures emerge in a competitive society?
What characterizes winners?

**Models:** Competition/Decline
Sudden Death Tournaments

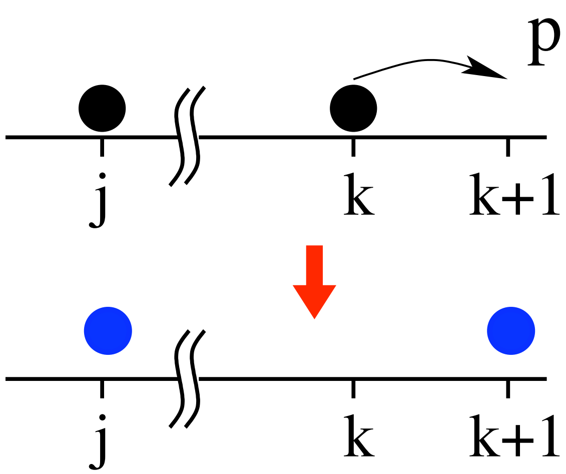
**Results:** *well-defined class structures*
*statistical properties of tournament winners*
*sports statistics applications*

# Competition/Decline Model

related work: Bonabeau et al (1995)

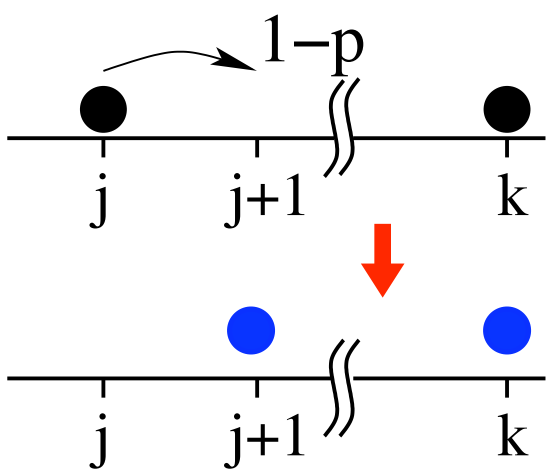
Each agent continuously competes with others to increase its fitness k≥0.

rate 1 & probability p:
$(j,k) \rightarrow (j,k+1)$ if $j \leq k$



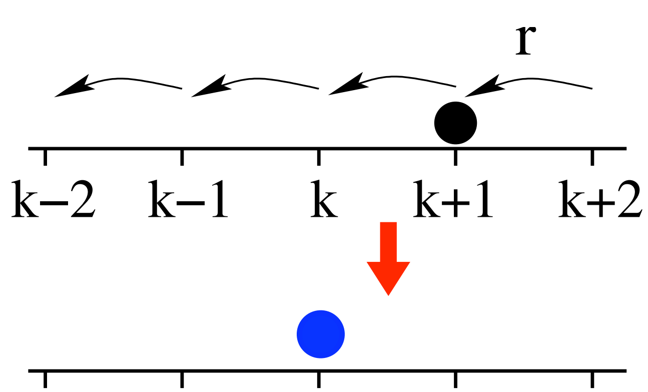

*stronger agent gains by competition*

rate 1 & probability q=1-p:
$(j,k) \rightarrow (j+1,k)$ if $j<k$




*weaker agent gains by competition*

rate r: $k \rightarrow k+1$




*decline in absence of stimulation*

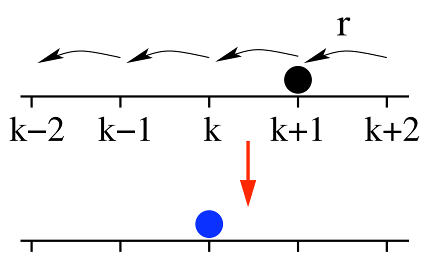
*What social structures emerge from these interactions?*

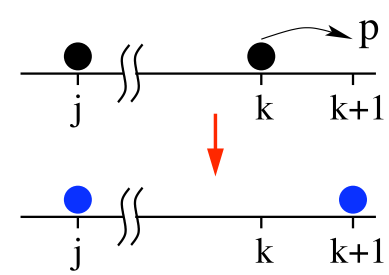
# Master Equation Description

$f_k(t) \equiv$ fraction of agents with fitness k at time t

$$F_k \equiv \sum_{j=0}^{k} f_j = \text{ fraction of agents with fitness} \leq \text{k}$$

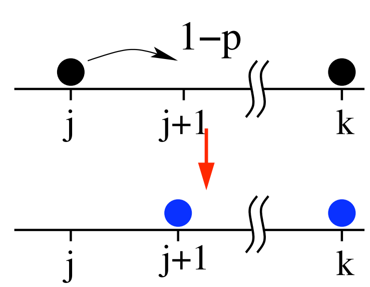
$$G_k \equiv \sum_{j=k+1}^{\infty} f_j = \text{ fraction of agents with fitness} > \text{k}$$




$$\frac{df_k}{dt} = \underbrace{r(f_{k+1} - f_k)}_{\text{decline}} + \underbrace{p(f_{k-1}F_{k-1} - f_kF_k)}_{\text{normal competition}}$$

$$+ \underbrace{q(f_{k-1}G_{k-1} - f_kG_k)}_{\text{upset competition}} + \underbrace{\frac{1}{2}(f_{k-1}^2 - f_k^2)}_{\text{ties}}$$



# The Cumulative Distribution

sum $$\frac{df_k}{dt} = r(f_{k+1} - f_k) + p(f_{k-1}F_{k-1} - f_k F_k) + q(f_{k-1}G_{k-1} - f_k G_k) + \frac{1}{2}(f_{k-1}^2 - f_k^2)$$

$$\frac{dF_k}{dt} = r(F_{k+1} - F_k) + (1-p)(F_{k-1} - F_k) + (p - 1/2)\left(F_{k-1}^2 - F_k^2\right)$$

*closed equation*

$$F_0 = 0, \quad F_\infty = 1;$$

*boundary conditions*

$$F_k(t=0) = 1, \quad k \geq 1$$

*initial condition*

## Partial information: mean fitness

$$\frac{d\langle k \rangle}{dt} = \frac{1}{2} - r(1 - f_0)$$

# Dynamical Behavior by Scaling Approach

master equation: $$\frac{dF_k}{dt} = r(F_{k+1} - F_k) + (1-p)(F_{k-1} - F_k) + (p - 1/2)\left(F_{k-1}^2 - F_k^2\right)$$

continuum limit: $$\frac{\partial F}{\partial t} = [p + r - 1 - (2p-1)F]\frac{\partial F}{\partial k}$$

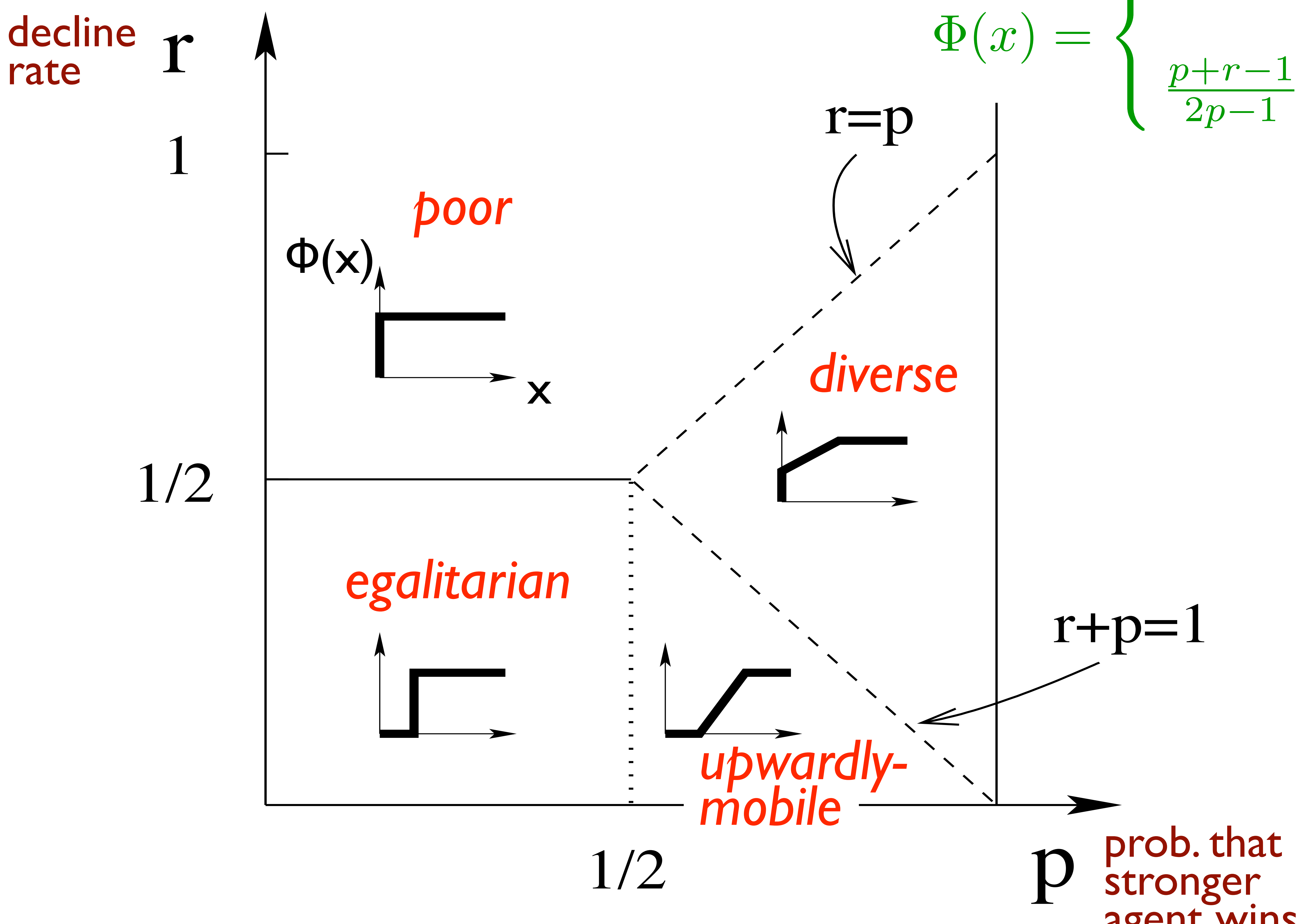
scaling ansatz: $$F_k(t) \sim \Phi(k/t) \qquad x \equiv k/t$$

$$\longrightarrow \quad [(p + r - 1 + x) - (2p-1)\Phi(x)]\,\frac{d\Phi}{dx} = 0$$

solutions: $$\Phi(x) = \begin{cases} \text{constant} \\ \frac{p+r-1}{2p-1} + \frac{x}{2p-1} \end{cases}$$

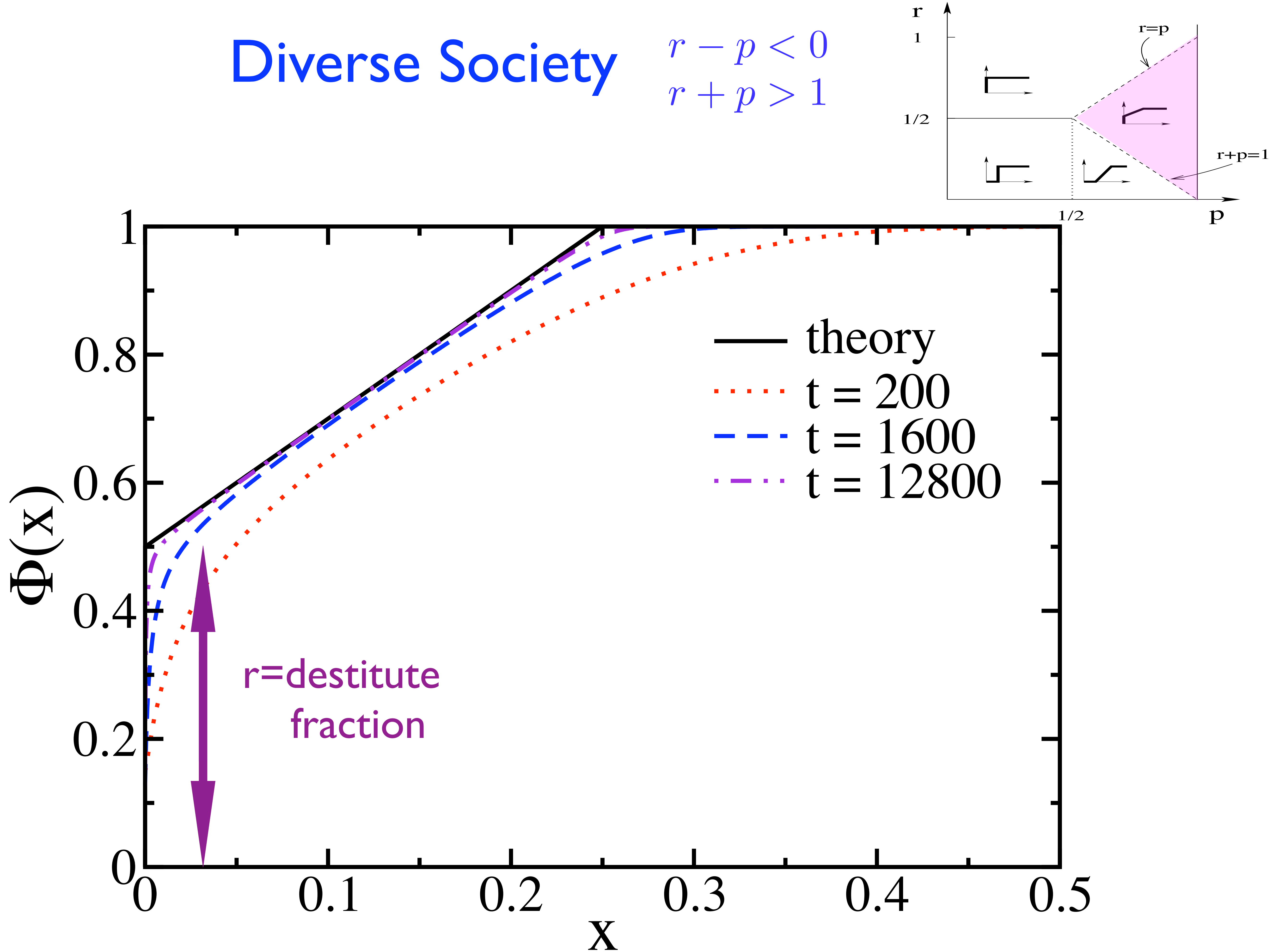
# Sketch of Scaling Behavior

$$\Phi(x) = \begin{cases} \text{constant} \\ \frac{p+r-1}{2p-1} + \frac{x}{2p-1} \end{cases}$$

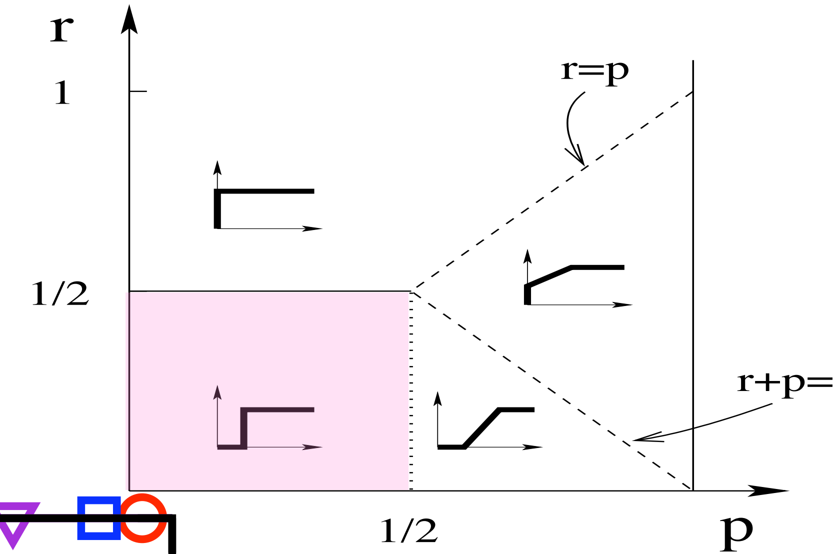


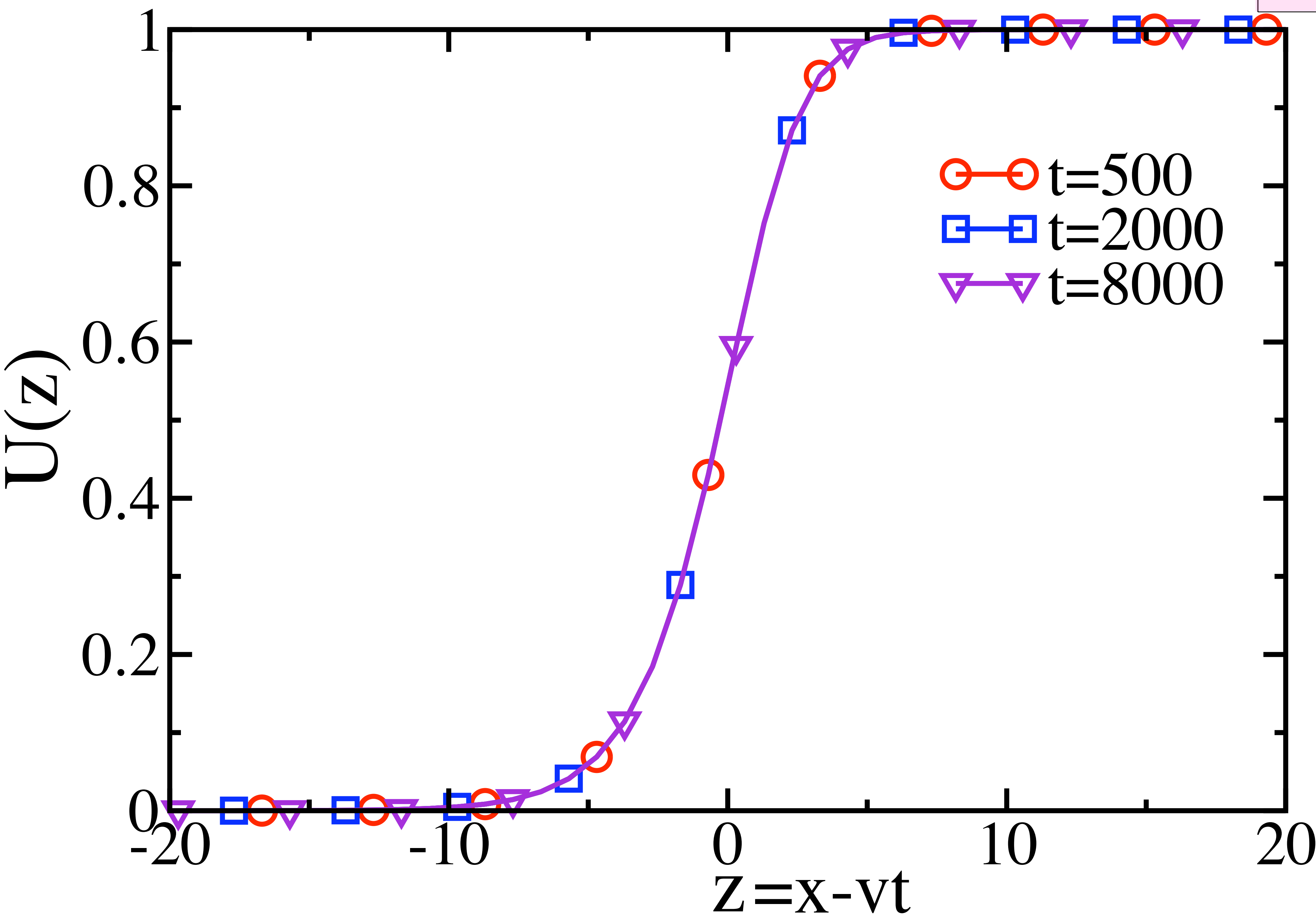
# Diverse Society

$r - p < 0$
$r + p > 1$



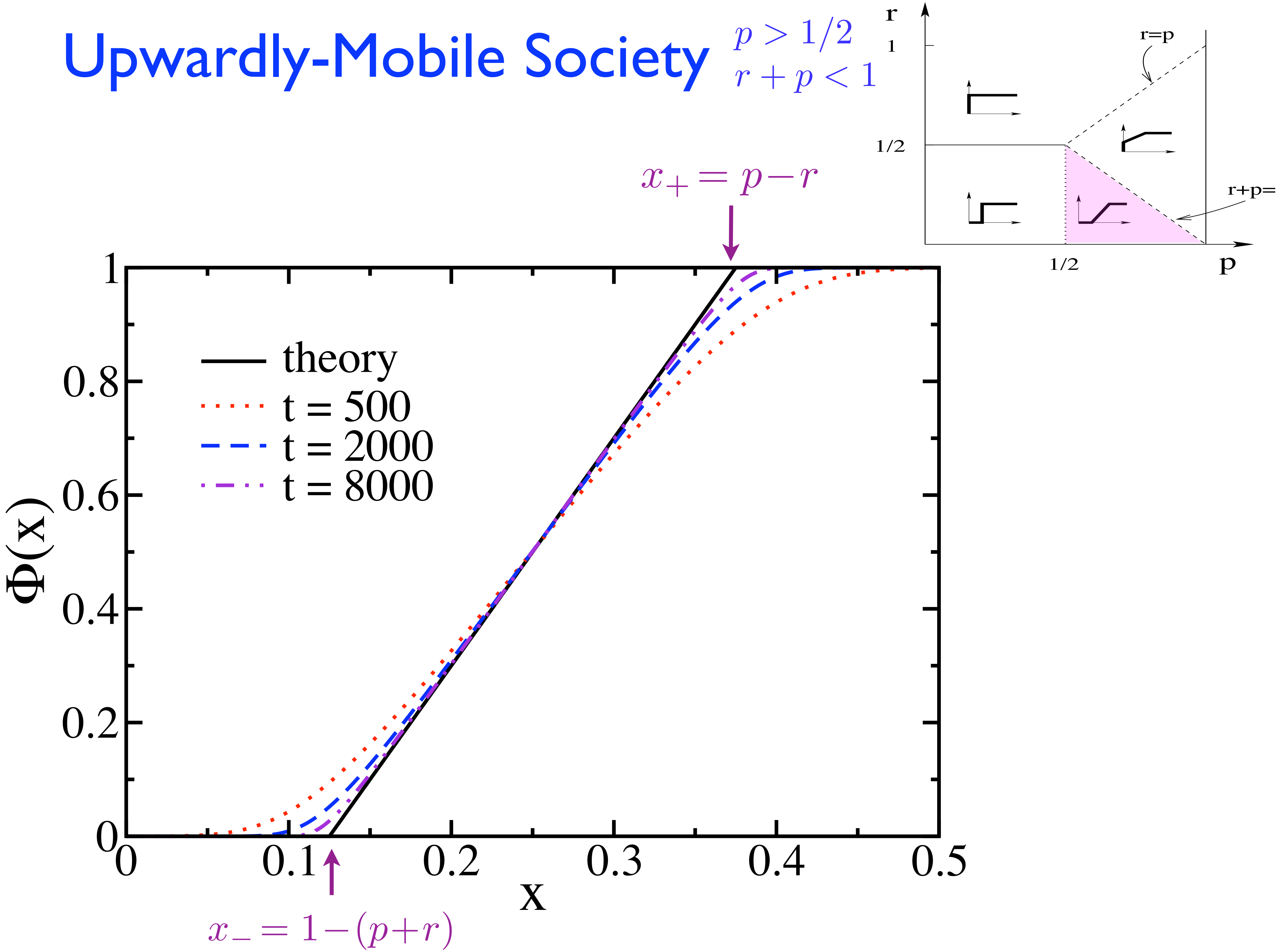
# Egalitarian Society $p, r < 1/2$



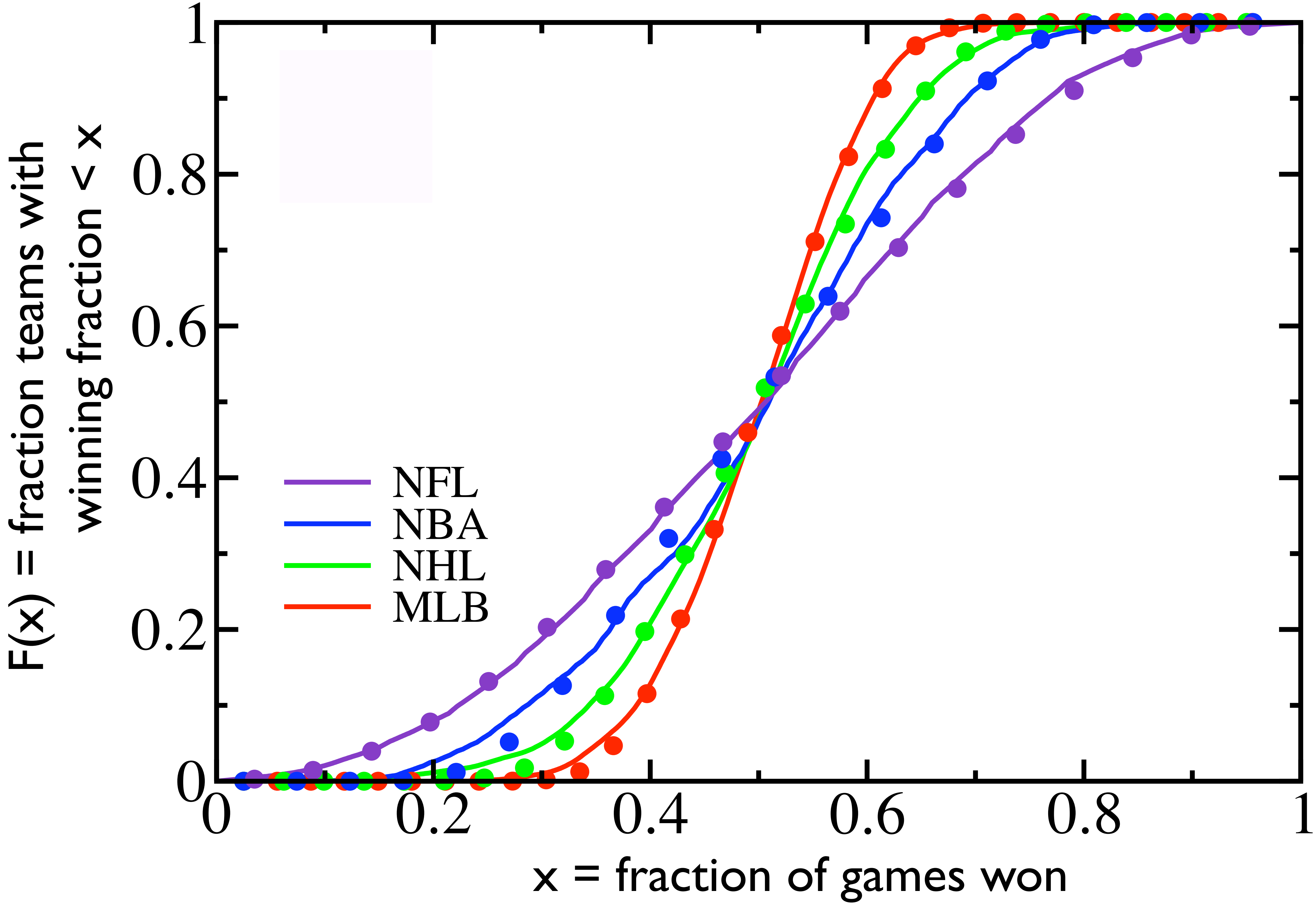



# Upwardly-Mobile Society

$p > 1/2$
$r + p < 1$

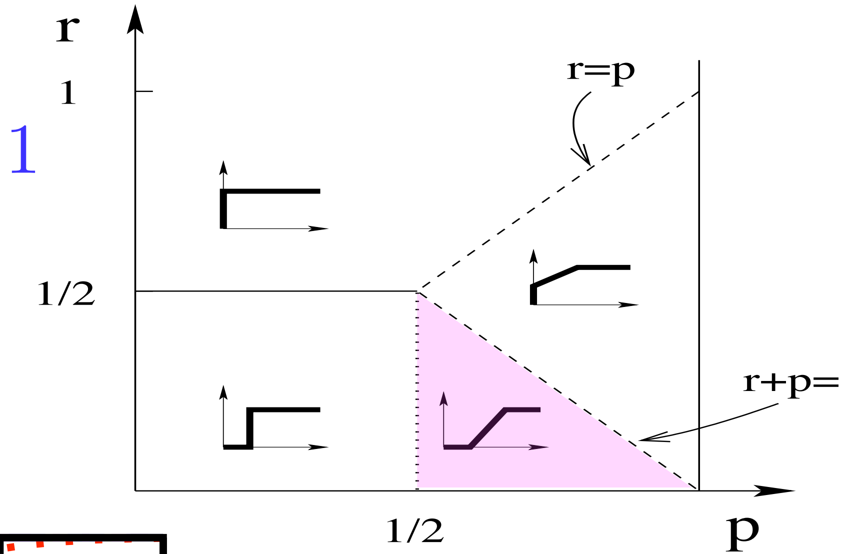


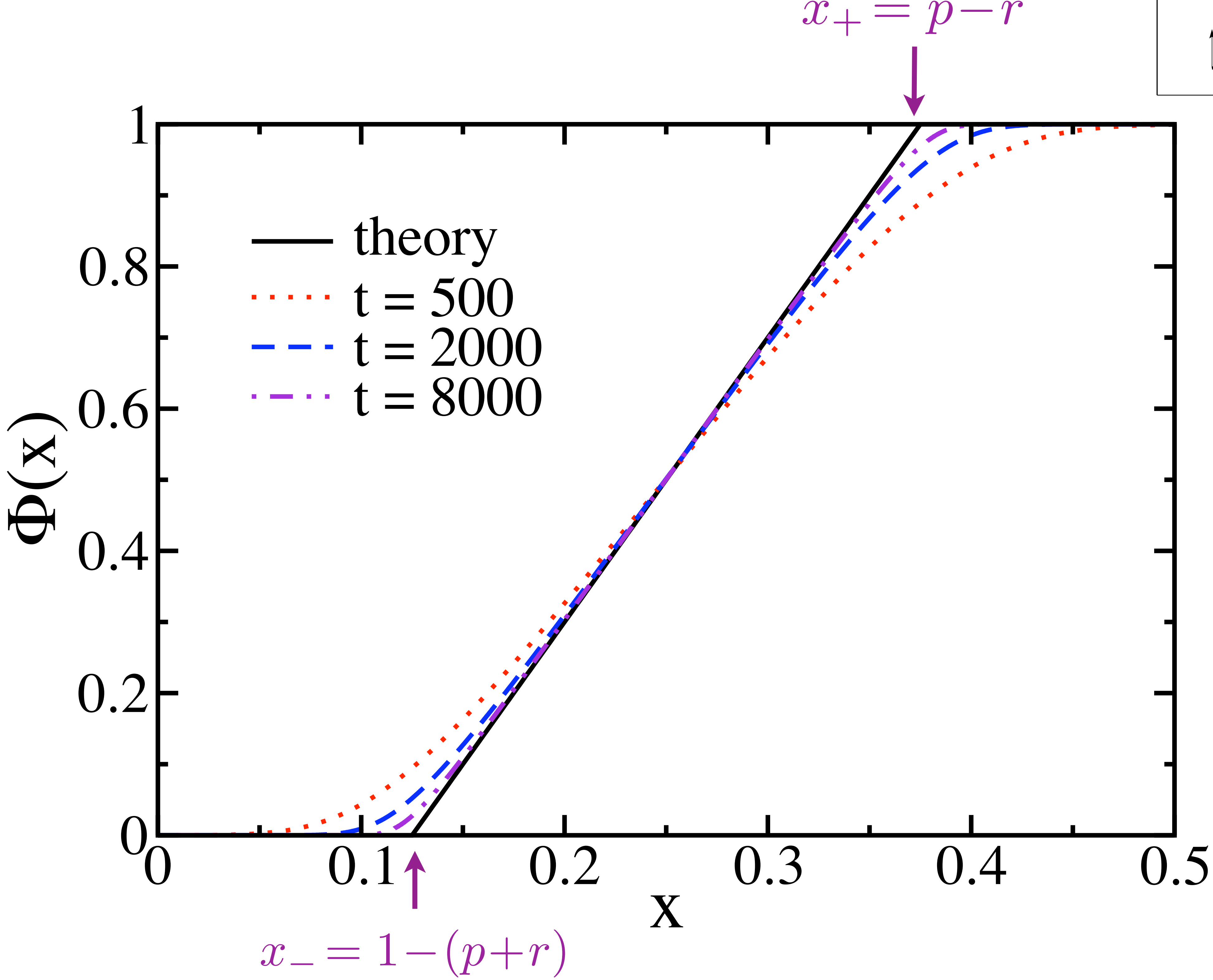
# Season-End Winning Fraction Distributions from Major Sports Leagues



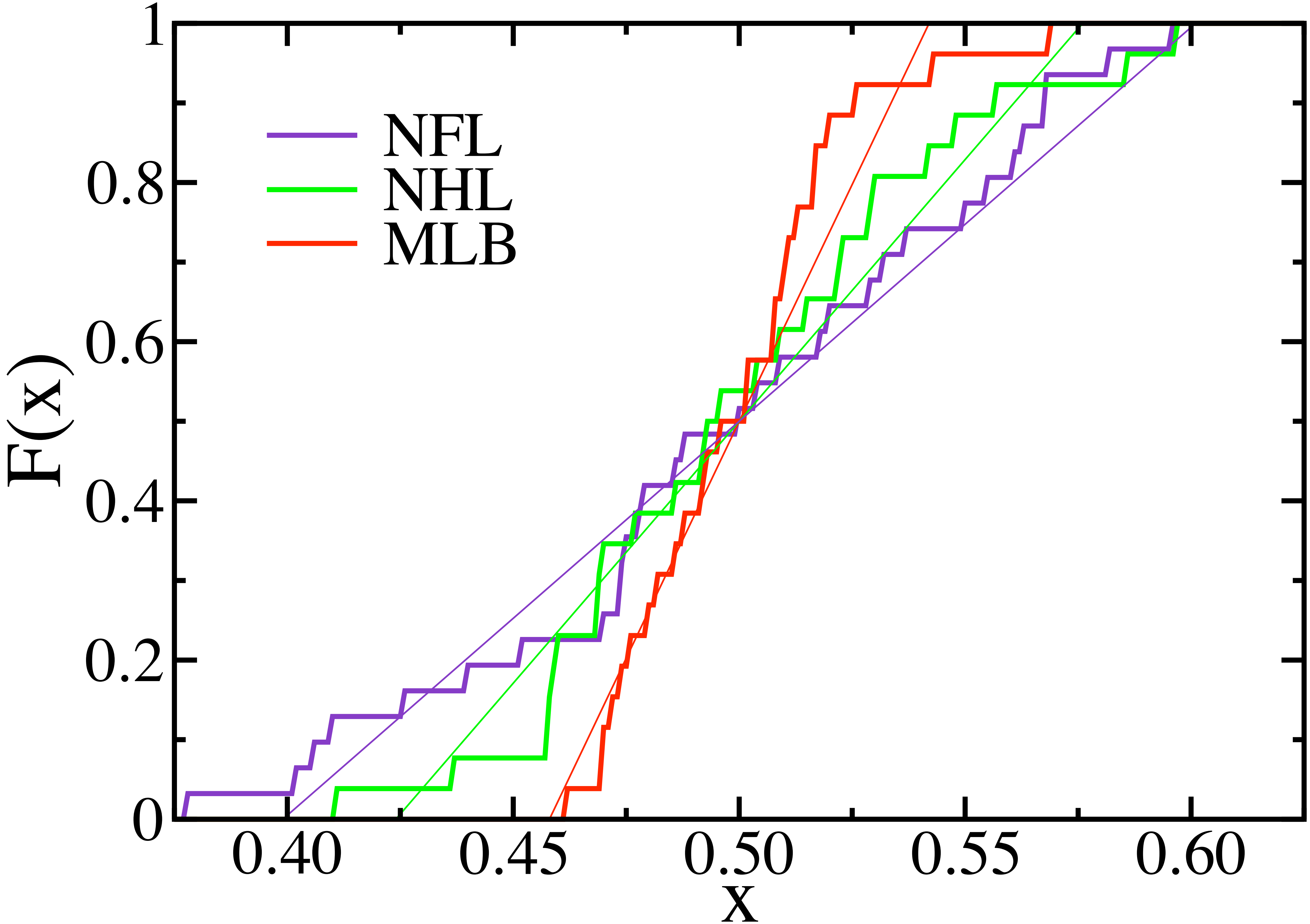
# Upwardly-Mobile Society

$p > 1/2$
$r + p < 1$



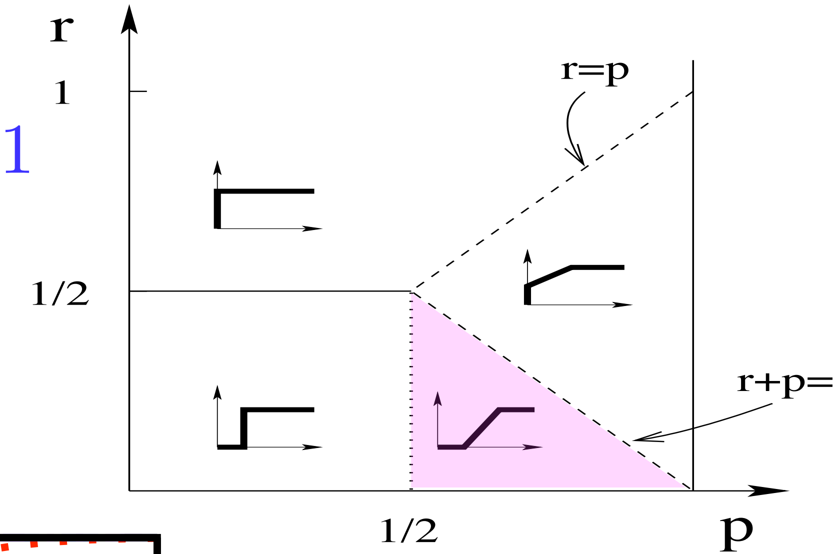



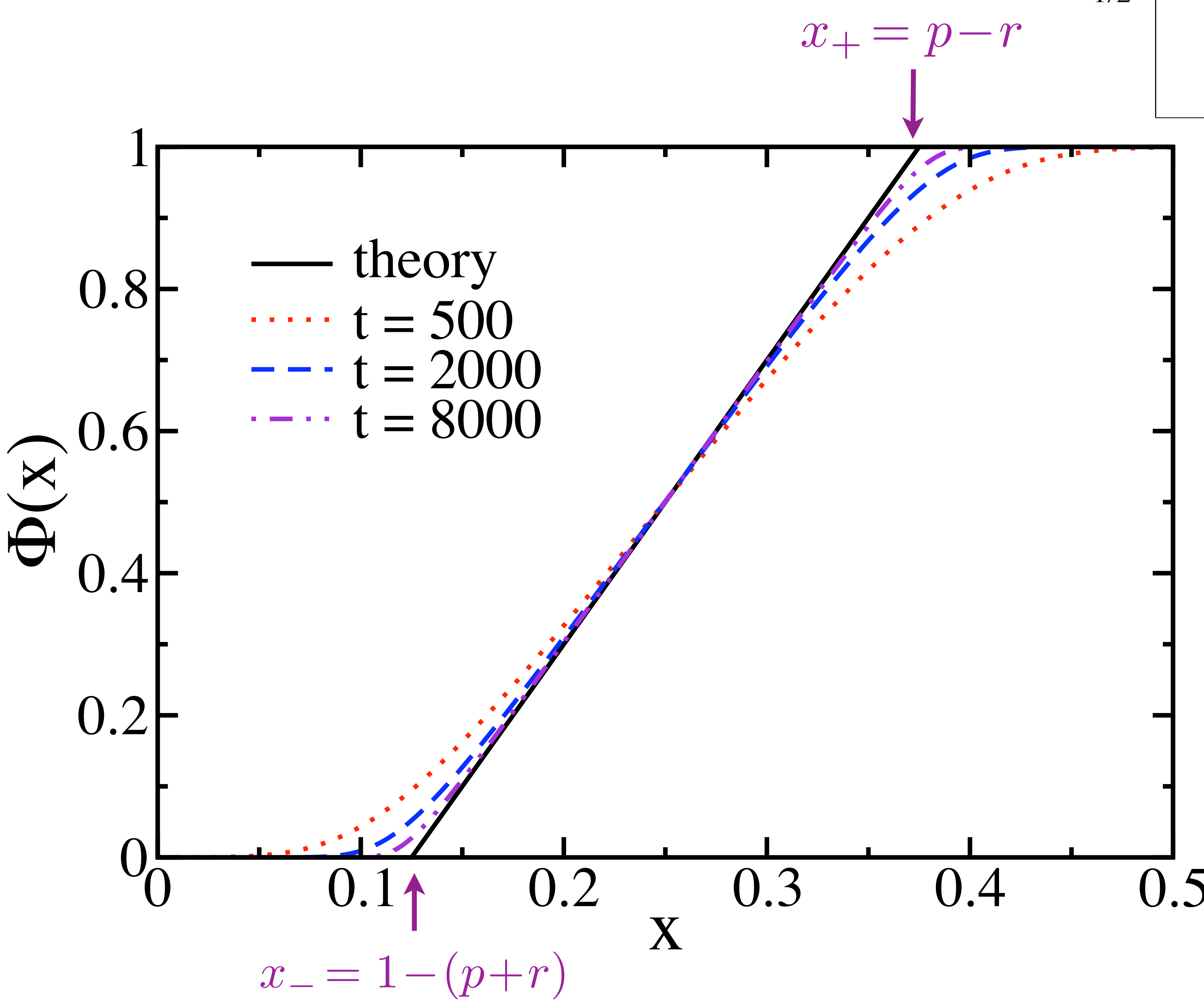
# All-Time Team Winning Fraction Distributions



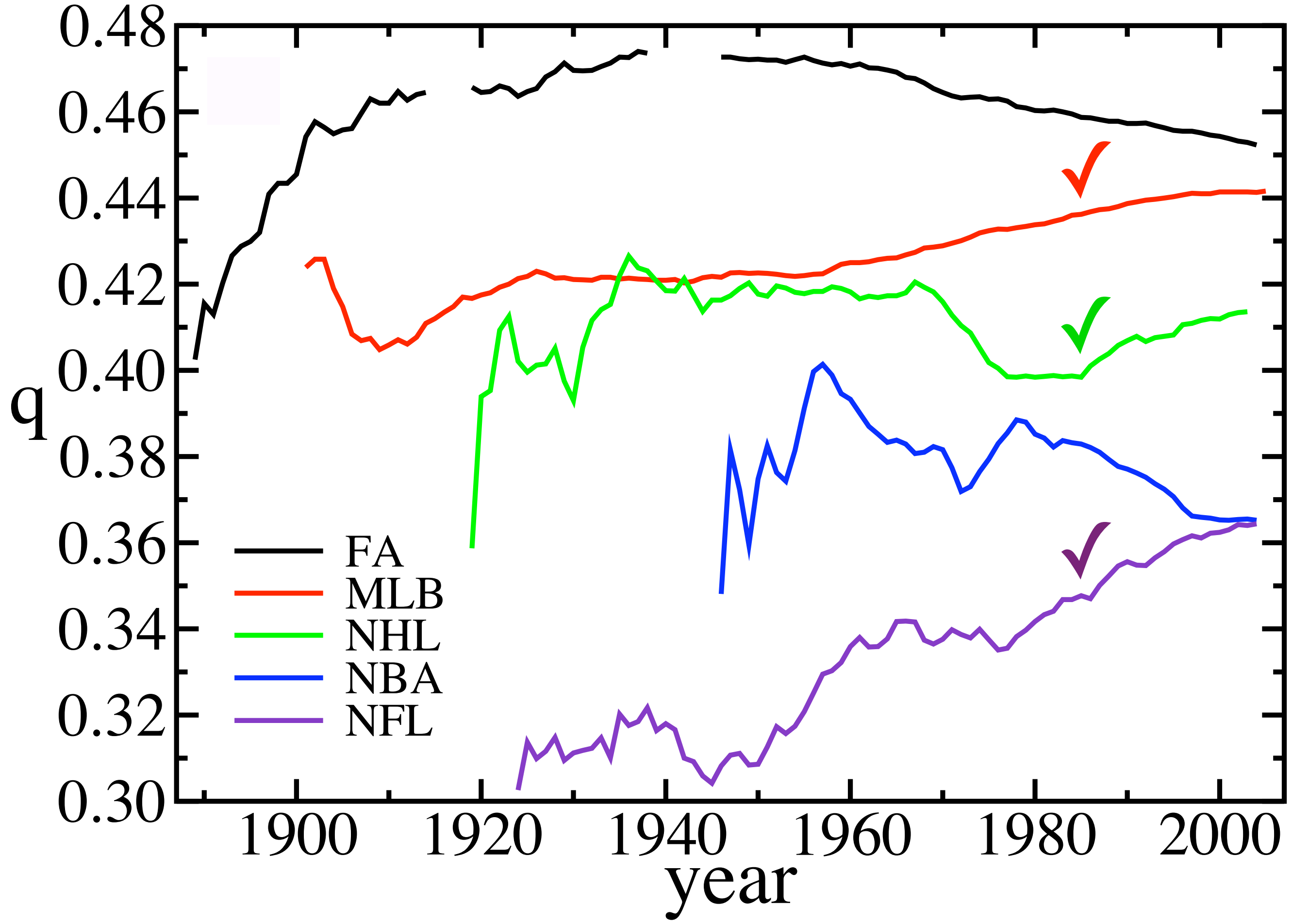
# Upwardly-Mobile Society

$p > 1/2$
$r + p < 1$



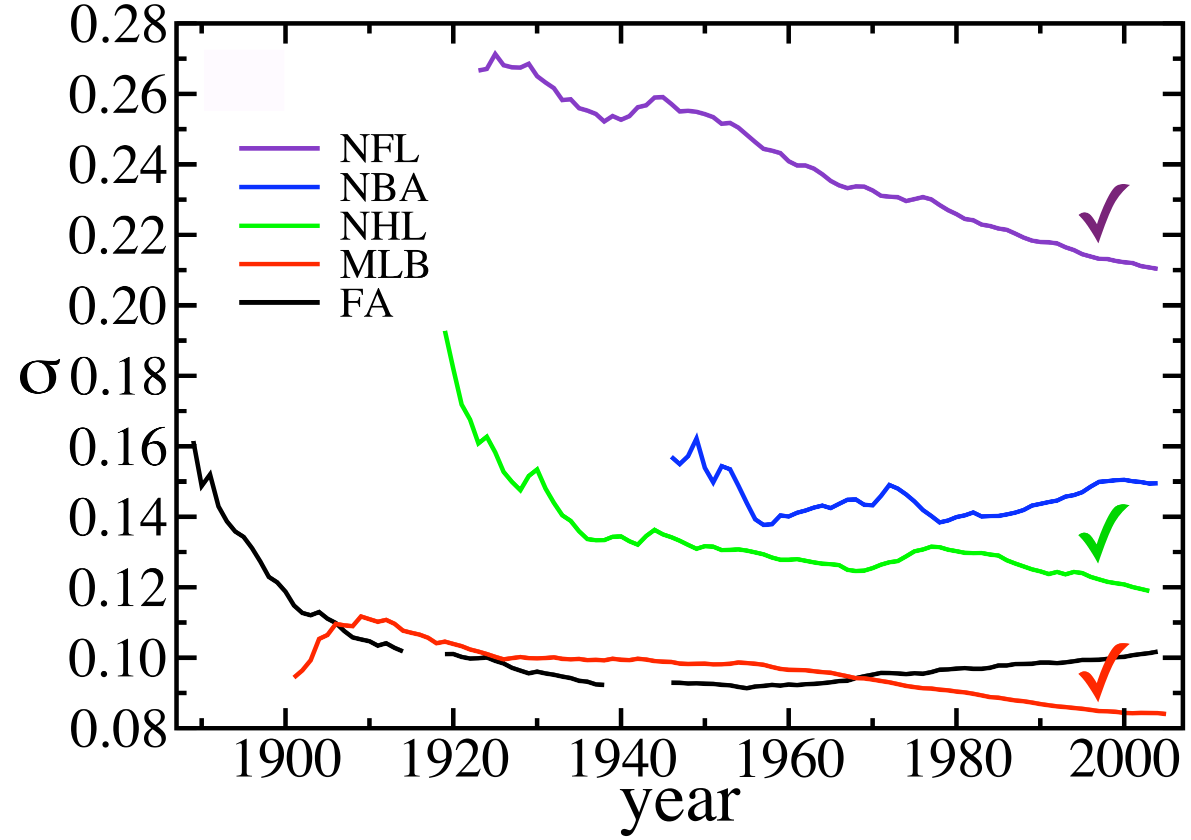



# Connection between Parity and Unpredictability




*increasing unpredictability*

*increasing parity*



# Dynamics of Sudden-Death Tournaments

fundamental variable: rank $x_k$

*lower number → better team*

evolution rule for 2 teams ranks $x_1$, $x_2$ $(x_1 < x_2)$:

$$(x_1, x_2) \to \begin{cases} x_1 & \text{with probability } 1-q; \\ x_2 & \text{with probability } q \end{cases}$$

*stronger team wins;*
*loser eliminated*

*weaker team wins (upset);*
*loser eliminated*

# Master Equation for Evolution of Rank Distribution

$f(x,t)dx \equiv \text{fraction of teams with rank} \in (x, x+dx)$

*(smaller x → better team)*

$$\frac{\partial f(x)}{\partial t} = -2p\, f(x) \int_0^x dy f(y) \;-\; 2q\, f(x) \int_x^\infty dy f(y)$$

**The cumulative distribution** $F(x) = \int_0^x f(y)\,dy$ **satisfies:**

$$\frac{\partial F}{\partial t} = (2q-1)F^2 - 2qcF$$

$$c(t) = \int_0^\infty f(x,t)\,dx = F(\infty)$$
$$= \text{fraction remaining teams}$$

Solve $\frac{\partial F}{\partial t} = (2q-1)F^2 - 2qcF$

$$F(x,t) \quad = \quad \frac{F_0(x)}{[1-F_0(x)](1+t)^{2q} + F_0(x)(1+t)}$$

$$= \quad \frac{x}{(1-x)(1+t)^{2q} + x(1+t)}$$

*for uniform initial rank distribution*

$$\to \quad t^{-1}\Phi\left(x\,t^{1-2q}\right) \qquad t \to \infty,\ x \to 0$$

$$\text{with } \Phi(z) = \frac{z}{1+z}$$

*scaling form*

*typical rank vs. time*

$$x \sim t^{-(1-2q)}$$

*rank of ultimate winner*

$$\longrightarrow x^* \sim N^{-(1-2q)}$$

# Parallel Dynamics (≠ Serial Dynamics!)

$g_N(x) \equiv$ rank distribution of winner in $N^{\text{th}}$ round

$G_N(x) = \int_0^x dy\, g_N(y)$ cumulative distribution in $N^{\text{th}}$ round

recursion for rank distribution:

$$g_2(x) = 2pg_1(x) \underbrace{[1 - G_1(x)]}_{\text{prob. weaker team}} + 2qg_1(x) \underbrace{G_1(x)}_{\text{stronger team}}$$

integrating gives:

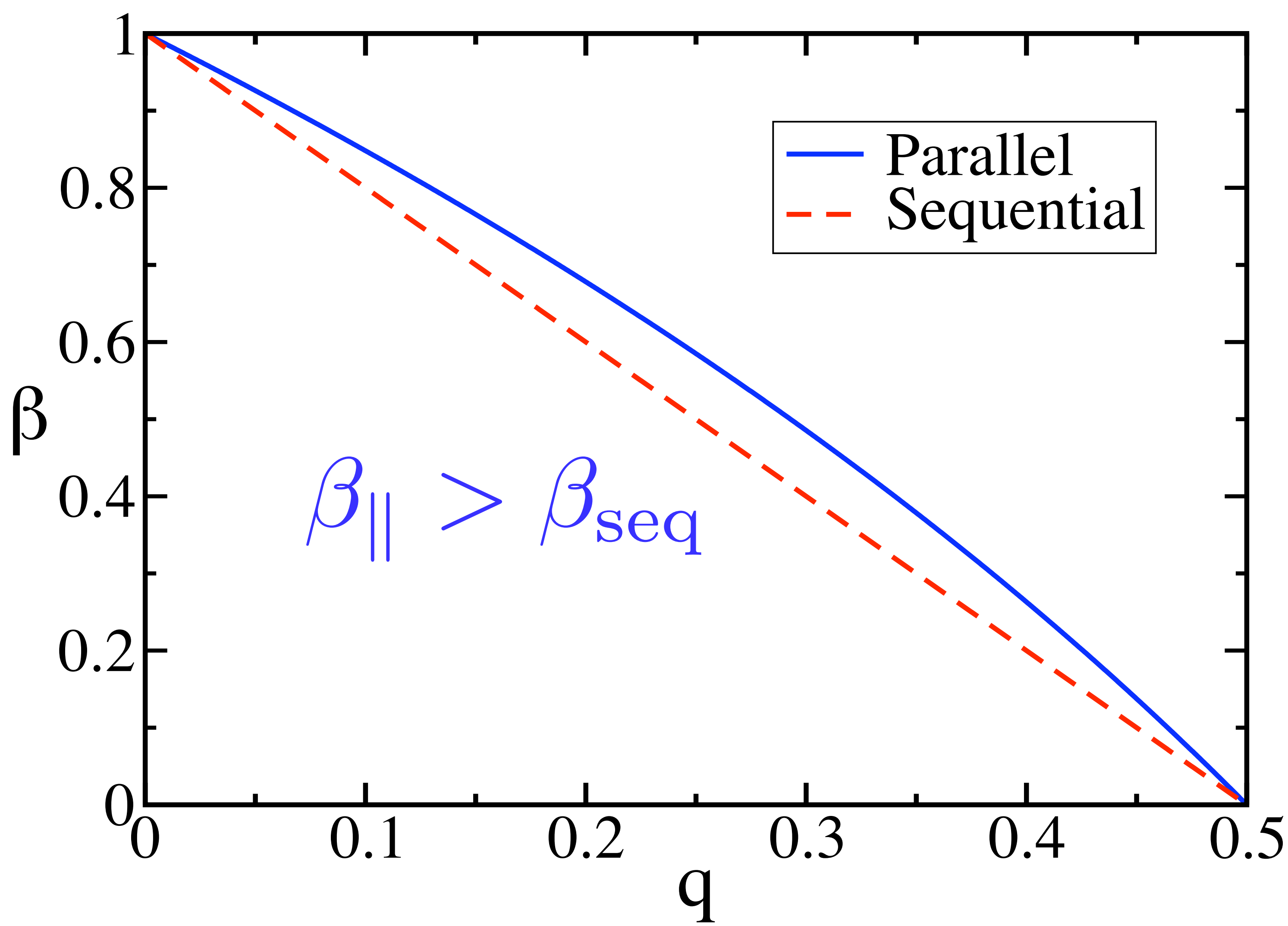
$$G_2(x) = 2pG_1(x) + (1-2p)[G_1(x)]^2$$

$$G_{2N}(x) = 2pG_N(x) + (1-2p)[G_N(x)]^2$$

# Asymptotic Solution

$$G_{2N}(x) = 2pG_N(x) + (1-2p)[G_N(x)]^2 \quad \rightarrow \quad G_{2^k}(x) \simeq (2p)^k\, x$$

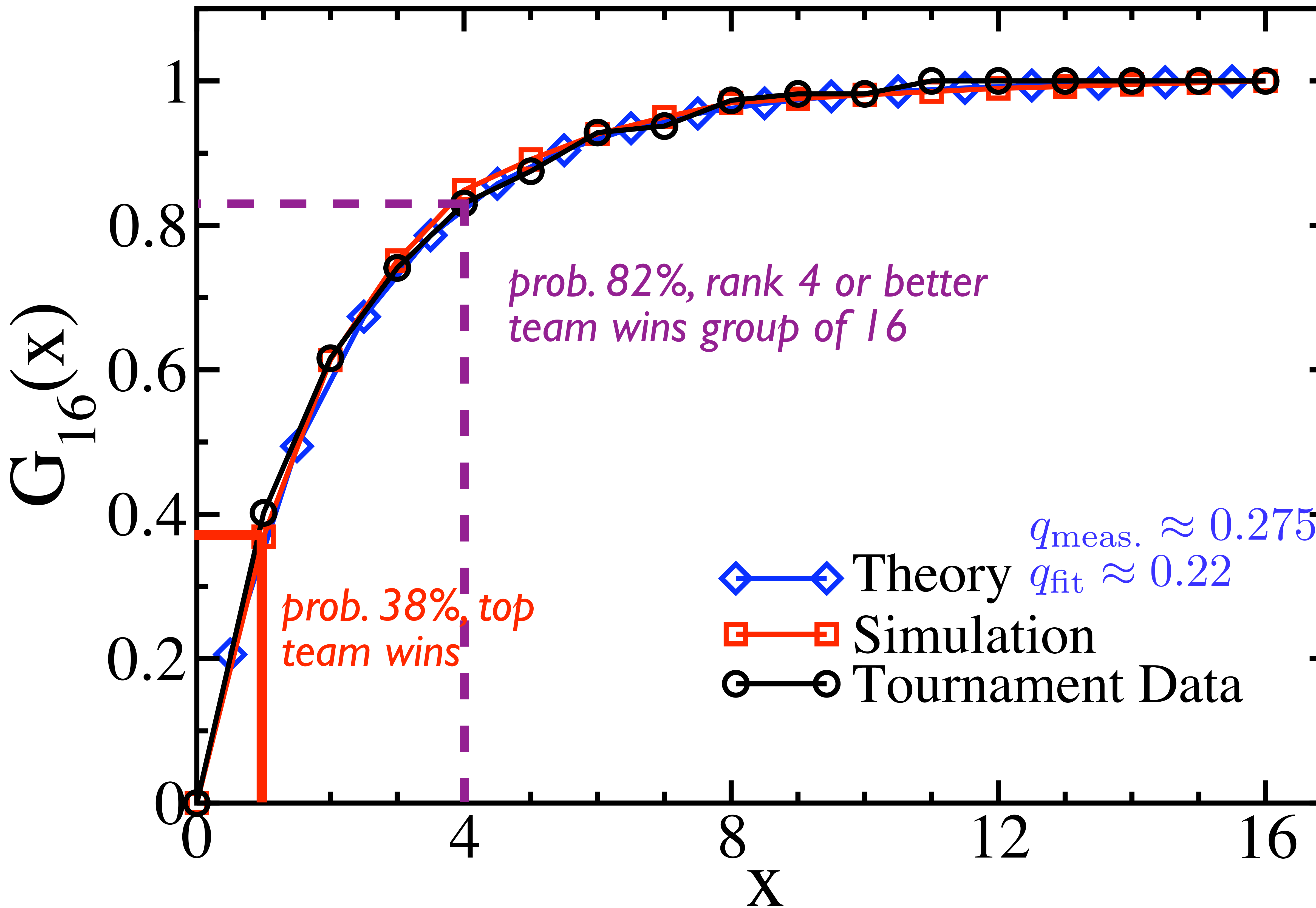
$$\text{Use } k = \frac{\ln N}{\ln 2} \quad \text{to give } G_N(x) \simeq N^{\beta_\parallel}\, x \quad \rightarrow \quad \beta_\parallel = 1 + \frac{\ln(1-q)}{\ln 2}$$




$$x \sim t^{-(1-2q)}$$

→ *weak teams less likely to survive parallel play*

# NCAA March Madness Results 1680 games 1979-2006



# Some Open Questions

What are the relative roles of intrinsic fitness versus luck?

What is the fate of a single agent? Can a rich person become poor?

What are the effects of symbiosis, deleterious competition, exogenous effects?

Is it possible to develop good betting strategies to exploit modeling & long-term sports statistics?