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Preface

You arrange a 7pm date at a local bistro. Your punctual date arrives at 6:55, waits until 7:05, concludes that you will not show up, and leaves. At 7:06, you saunter in – "just a few minutes" after 7 (see Cover). You assume that you arrived first and wait for your date. The wait drags on and on. "What's going on?" you think to yourself. By 9pm, you conclude that you were stood up, return home, and call to make amends. You explain, "I arrived around 7 and waited 2 hours! My probability of being at the bistro between 7 and 9pm, $P(\text{bistro}, t)$, was nearly one! How did we miss each other?" Your date replies, "I don't care about your *occupation* probability. What mattered was your *first-passage* probability, $F(\text{bistro}, t)$, which was zero at 7pm. GOOD BYE!" Click!

The moral of this juvenile parable is that first passage underlies many stochastic processes, in which the event, such as a dinner date, a chemical reaction, the firing of a neuron, or the triggering of a stock option, relies on a variable reaching a specified value *for the first time*. In spite of the wide applicability of first-passage phenomena (or perhaps because of it), there does not seem to be a pedagogical source on this topic. For those with a serious interest, essential information is scattered and presented at diverse technical levels. In my attempts to learn the subject, I also encountered the proverbial conundrum that a fundamental result is "well-known to (the vanishingly small subset of) those who know it well".

In response to this frustration, I attempt to give a unified presentation of first-passage processes and illustrate some of its beautiful and fundamental consequences. My goal is to help those with modest backgrounds learn essential results quickly. The intended audience is physicists, chemists, mathematicians, engineers, and other quantitative scientists. The technical level should be accessible to the motivated graduate student.

My literary inspirations for this book include *Random Walks and Electric Networks*, by P. G. Doyle and J. L. Snell (Carus Mathematical Monographs #22, Mathematical Association of America, Washington, D. C.,1984), which cogently describes the relation between random walks and electrical networks, and *A Primer on Diffusion Problems*, by R. Ghez (Wiley, 1988) which gives a nice exposition of solutions to physically-motivated diffusion problems. This book is meant to complement classic monographs, such as *An Introduction to Probability Theory and its Applications*, by W. Feller (Wiley, New York, 1968), *Aspects and Application of the Random Walks*, by G. H. Weiss, (North-Holland, Amsterdam, 1996), and *Stochastic Processes in Physics and Chemistry*, by N. G. van Kampen (North-Holland, Amsterdam, 1997). Each of these very worthwhile books discusses first-passage phenomena, but secondarily rather than as a comprehensive overview.

I begin with fundamental background in Chap. 1 and outline the relation between occupation and first-passage probabilities, as well as the connection between first passage and electrostatics. Many familiar results from electrostatics can be easily adapted to give first-passage properties in the same geometry. In Chap. 2, I discuss first passage in a one-dimensional interval. This provides a simple laboratory for answering basic questions, such as: What is the probability that a diffusing particle eventually exits at either end? How long does it take to exit? These problems are solved both by direct approaches and by developing the electrostatic equivalence. Chapter 3 treats first passage in a semi-infinite interval both by standard approaches and by the familiar image method. I also discuss surprising consequences of the basic dichotomy between certain return to the starting point and infinite mean return time.

Chapter 4 is devoted to illustrations of the basic theory. I discuss neuron dynamics, realizations of self-organized criticality, and the dynamics of spin systems. These all have the feature that they can be viewed as first-passage processes in one dimension. I also treat stochastic resonant escape from fluctuating and inhomogeneous media, for which the time-independent electrostatic formalism provides a relatively easy way to solve for mean first-passage times. Finally, I discuss the survival of a diffusing particle in a growing "cage" and near a moving "cliff", where particularly rich behavior arises when diffusion and the motion of the boundary have the same time dependence.

In Chap. 5, I turn to first passage on branched, self-similar structures. I emphasize self-similar systems because this feature allows us to solve for the first-passage probability by renormalization. Another essential feature of branched systems is the competition between transport along the "backbone" from source to sink and detours along sidebranches. I give examples that illustrate this basic competition and the transition from scaling, in which a single time scale accounts for all moments of the first-passage time, to multiscaling, in which each moment is governed by a different time scale.

I then treat spherically-symmetric geometries in Chap. 6 and discuss basic applications, such as efficient simulations of diffusion-limited aggregation and the Smoluchowski chemical reaction rate. First passage in wedge and conical domains are presented in Chap. 7. I discuss how the wedge geometry can be solved elegantly by the mapping to electrostatics and conformal transformations. These systems provide the kernel for understanding the main topic of Chap. 8, namely, the kinetics of one-dimensional diffusion-controlled reactions. This includes trapping, the reactions among three diffusing particles on the line, as well as basic bimolecular reactions, including capture $p + P \to P$, annihilation $A + A \to 0$, coalescence $A + A \to A$, and aggregation $A_i + A_j \to A_{i+j}$. The chapter ends with a brief treatment of ballistic annihilation.

A large fraction of this book discusses either classical first-passage properties or results about first passage from contemporary literature, but with some snippets of new results sprinkled throughout. However, several topics are either significant extensions of published results or are original. This includes the time-integrated formalism to compute the first-passage time in fluctuating systems (Section 4.5), aspects of survival in an expanding interval (Section 4.7), return probabilities on the hierarchical tree and homogeneous comb (Sections 5.3 & Subsection 5.4.2), the first-passage probability on the hierarchical blob (Section 5.5), and reactions of three diffusing particles on the line (Section 8.3).

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Even in the final stages of writing, I am acutely aware of many shortcomings in my presentation. If I were to repair them all, I might never finish. This book is still work "in progress" and I look forward to receiving your corrections, criticisms, and suggestions for improvements (redner@bu.edu).

Finally and most importantly, I thank my family for their love and constant support and for affectionately tolerating me while I was writing this book.