Recent Progress and Current Puzzles in Percolation

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ABSTRACT

The basic physical phenomena of percolation are reviewed within the context of the modern theory of critical phenomena. The connection between percolation and the Potts model, a statistical mechanical model of ferromagnetism, is discussed. Recent advances in calculating critical exponents by position-space renormalization group methods are also described. Several open questions are also raised, including the nature of cluster structure and transport near the percolation threshold and the anomalous geometrical properties of self-similar structures.

1. Introduction

The general problem of understanding the properties of disordered media is an area of considerable importance and interest. An important conceptual advance for describing such systems is the percolation model, first introduced by Broadbent and Hammersley in 1957 (for recent reviews and extensive references, see e.g., Stauffer 1979, Essam 1980). This is an idealized model which appears to capture the essential physical mechanism underlying many important features of random media, that of connectivity.

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In order to gain an intuition for percolation, it is helpful to begin with a pictorial account. Consider bond percolation on the square lattice defined by randomly occupying the edges of the lattice at a given probability p. Figure 1(a) shows a 20×20 section of the square lattice whose edges are occupied with $p = 0.35$. From a geometrical point of view, there are two points that deserve emphasis: (i) Only small clusters up to a length scale denoted by $\xi(p)$, the correlation length, occur. The number of clusters with radii $> \xi$ is exponentially small. (ii) The clusters are disconnected; it is not possible to find a continuous path that traverses the sample. This situation is analogous to the disordered phase of a ferromagnet.

In Fig. 1(b), a sample at $p = 0.65$ is shown. There now exists one very large connected cluster which traverses across the lattice. This connected state is analogous to the ordered phase in the ferromagnetic transition. Notice also that the largest cluster has a strong propensity to absorb finite clusters as $p \to 1$. The correlation length, defined as the characteristic length scale of finite clusters, therefore vanishes in this limit.

Intermediate to the situations depicted in figures 1(a) and 1(b) is the percolation threshold. For the square lattice this is known by duality arguments to occur $p = p_c = 1/2$. An example of such a critical state is shown in Fig. 1(c), on which a percolating path is indicated. In particular, notice that there exist clusters of all length scales (up to the linear dimension of the lattice) and ξ diverges. This singular behaviour of ξ turns out to be crucial in understanding the physical properties of random media.

The connectivity transition at the percolation threshold is second order so that various geometrical properties exhibit power law singularities as $p \to p_c$. One example is the correlation length, ξ, which diverges as

$$(p - p_c)^{-\nu}, \tag{1}$$

as $p \to p_c$ (Fig. 2). A second important quantity is $P(p)$, the percolation probability, defined as the probability that a randomly chosen bond belongs to the infinite cluster. For $p < p_c$, there is no infinite cluster and $P = 0$. Above p_c, this probability is finite and it vanishes as

$$(p - p_c)^{\beta}, \tag{2}$$

as $p \downarrow p_c$.

A central tenet in the modern theory of critical phenomena is that critical exponents are generally <u>universal</u> (see e.g., Ma 1976, Pfeuty and Toulouse 1977). Since the transition involves phenomena on all length scales, local details of a model, such as lattice structure, should be irrelevant. For percolation, the exponents do depend on only one parameter, the spatial dimension d of the underlying lattice. The dependence of the exponents on d is the key to understanding the critical phenomena of a given model. As Fig. 3 indicates, there are three regimes of interest. First, above an upper critical dimension d_c, spatial fluctuations occuring in a system near its phase transition may be neglected. Simple analytic mean-field theories are accurate in this regime, and one finds exponents independent of d. Between d_c and d_L, the lower critical dimension, fluctuations are important and they cause exponents to become dependent on dimension. Considerable effort has been devoted to the calculation of critical exponents in this regime by a variety of techniques. Finally, below d_L, the lattice does not possess sufficient topological connections to propogate order or connectivity, and no phase transition is possible.

While percolation is an extremely intriguing and appealing geometrical problem, the model has many immediate applications to disordered physical problems whose properties are mediated by connectivity. Several examples are summarized in table 1.

Table 1: Connection between percolation and physical problems.

bond	system described	calculable properties
spring	sol-gel transition	viscosity, bulk and shear moduli
resistor	random resistor network	conductivity of metal-insulator mixtures
hopping rate	transport in random media	diffusion coefficients

2. Recent Progress

(a) Potts Model Formulation

An important mapping between percolation and the Potts model of ferromagentism was elucidated by Kasteleyn and Fortuin (1969). This was an important advance as it gave a firm basis for the description of percolation as a second order phase transition, and the Potts model served as a starting point for many renormalization group calculations of critical behaviour.

To define the Potts model, consider a regular lattice with spins σ_i at each site, which can assume any one of s values $1,2,\ldots s$. Geometrically one may think of the spins as pointing from the center to one of the s vertices of a symmetric tetrahedron embedded in an $(s-1)$ dimensional space. Nearest-neighbour pairs of spins have a ferromagnetic interaction which is proportional to the dot product between the two spin vectors. The Hamiltonian or energy of the entire lattice can be written as

$$\mathcal{H} = - J \sum_{\langle i,j \rangle} (\delta_{\sigma_i \sigma_j} - 1), \tag{3}$$

where the sum runs over all nearest-neighbour pairs of spins σ_i and σ_j. To make a correspondence with percolation, consider a high-temperature expansion for the partition function Z. After several elementary manipulations one obtains,

$$\begin{aligned} Z &= \mathrm{Tr}\, e^{-\beta \mathcal{H}} \\ &= \mathrm{Tr} \prod_{\langle i,j \rangle} e^{\beta J(\delta_{\sigma_i \sigma_j} - 1)} \\ &= \mathrm{Tr} \prod_{\langle i,j \rangle} (e^{-\beta J} + (1 - e^{-\beta J})\delta_{\sigma_i \sigma_j}). \end{aligned} \tag{4}$$

Upon expanding out the product, the partition function is converted into a graphical expansion in which the factor $(1 - e^{-\beta J})\delta_{\sigma_i \sigma_j}$ represents a bond between sites i and j. Figure 4 shows a typical term in this expansion; each solid bond represents a factor $p = 1 - e^{-\beta J}$, and each site represents a factor of σ_i. When the trace over all spin states is performed, the product of delta functions forces the spin states in each connected cluster to be identical. There is one more sum than delta function constraints for every cluster, so that each one is weighted by a factor of s. With these results, the partition function becomes

$$Z = \sum_{\text{clusters}} p^{N_B} q^{\mathcal{N}-N_B} s^{N_c} = \langle s^{N_c} \rangle , \tag{5}$$

where $\mathcal{N}$ is the total number of bonds in the lattice, N_B is the number of occupied bonds, N_c is the number of clusters, and $\langle \ldots \rangle$ denotes a configurational average. In the limit $s \rightarrow 1$, the free energy becomes $\langle N_c \rangle$, the average number of clusters. Various moments of this cluster generating function then yield all the physically important quantities in percolation (Wu 1982).

Another intriguing mapping arises by choosing the interaction strength J proportional to s and taking the limit $s \rightarrow 0$ (Fortuin and Kastelyn 1972, Stephen 1976). In this limit, one obtains the generating function for all spanning trees, defined as connected subgraphs of the lattice which visit every site and which contains no closed loops. Spanning trees arise naturally in the solution to Kirchhoff's laws (see e.g., Wu 1982); consequently the zero-state Potts model may be used to describe random resistor networks.

In summary, the Potts model provides a concrete statistical mechanical model for percolation, and it may be used as a starting point for further developments. By making a correspondence between the Potts model and certain exactly soluble statistical mechanical models, several conjectures for the exponents of percolation in two dimensions have been proposed which appear to be exact (den Nijs 1979, Pearson 1980). The construction of mappings between apparently unrelated models is a very elegant tool for obtaining critical behaviour.

(b) Position Space Renormalization Group Approach

Of the many methods developed to study phase transition phenomena, the position space renormalization group (PSRG) approach has proved to be of great utility. The primary advantage of the PSRG is that it is inherently constructed to treat the formidable problem of all length scales contributing to critical phenomena. In contrast, methods such as series expansions treat this feature only perturbatively, and therefore converge slowly near the critical point.

The basic idea underlying the PSRG is illustrated in Fig. 5 where a schematic picture of finite areas of three infinite random lattices at $p < p_c$, $p = p_c$, and $p > p_c$ are shown on the top row. Now imagine "zooming away" by a factor of 2 (rescaling all lengths by a factor of 2), and then turning down the contrast by a factor of 2 (summing out degrees of freedom at short distance scales) to arrive at the situation depicted in the bottom row of the figure. The change in cluster sizes under length rescaling may be represented by a "renormalization" of p to a new value p'. The functional relation $p' = R(p)$ is called a renormalization transformation. Under rescaling, a lattice at $p < p_c$ will renormalize to a new probability p' which is less than the initial probability p. The value of p is eventually renormalized to zero as the rescaling is repeated indefnitely. This represents a stable fixed point of the transformation. On the other hand, for $p > p_c$, the rescaling procedure iterated an infinite number of times leads to a stable fixed point at $p = 1$ as finite "holes" eventually renormalize away.

The situation at $p = p_c$ is quite different. There is no longer any characteristic length scale, so that the system appears statistically identical both before and after rescaling. The critical point represents an unstable fixed point of the renormalization transformation. This "self-similarity" of a critical system under rescaling is the source of many interesting physical phenomena. The three fixed points just discussed represent the possible final states of repeated lattice rescalings: $p = 0$, $p = 1$, or $p = p_c$, the latter being the critical system.

To obtain exponents through renormalization, note that if all lengths are rescaled by a factor of b then

$$\xi(\Delta p) = \xi'(\Delta p')/b, \tag{6}$$

where $\Delta p = p - p_c$ is the deviation of the concentration from its critical value, and $\Delta p'$ is the renormalized value of the deviation. Furthermore, from Eq. (1) we had $\xi \sim (\Delta p)^{-\nu}$. Combining with (6) then leads to,

$$\nu = \ln b / \ln (dp'/dp)|_{p=p^*}. \tag{7}$$

where p* is the unstable fixed point of the renormalization transformation, R(p). It is an approximation for the critical probability, p_c. At this fixed point, critical exponents may be computed from the linearization of R(p) about p*.

A particularly simple way of constructing a renormalization transformation was introduced by Reynolds et al (1977). For bond percolation on the square lattice, the lattice is broken up into square b x b cells which tesselate the lattice as indicated in Fig. 6. Upon rescaling, each such cell will map into a smaller b′ x b′ cell of the same topology. The simplest approximation is to choose b = 2 and b′ = 1. Since the basic physics of percolation is connectivity, a plausible definition for the renormalization transformation is to rescale all configurations which traverse the b x b cell to an occupied bond on the 1 x 1 cell. Hence p′ is simply the total probability of spanning the relevant portion of the cell (bottom row of the figure). For the b = 2 to b′ = 1 rescaling, this is the probability of getting from either of the bottom two sites of the cell to either of the top two sites,

$$p' = R(p) = 2p^2 + 2p^3 - 5p^4 + 2p^5 \tag{5}$$

The unstable fixed point of Eq. (5) occurs at p* = 1/2, which gives the exact value of p_c (this stems from the self-duality of cells chosen to construct the transformation). In addition, Eq. (7) gives $\nu \cong 1.428$, which is a reasonable approximation to the currently accepted value of $\nu = 4/3$ (den Nijs 1979), in view of the simplicity of the calculation.

The power of the cell renormalization is its simplicity and relative accuracy for a fixed amount of calculational effort. Furthermore, it is possible to systematically reduce the errors introduced in the approximation by considering larger cells (Reynolds et al 1980). By this extension, the cell PSRG yields an accuracy which is comparable to other modern statistical mechanical methods. Finally, the method can be generalized to treat conductivity of random resistor networks (Bernasconi 1978), self-avoiding walks (de Queiroz and Chaves 1980, Redner and Reynolds 1981), lattice animals (Family 1980), directed percolation problems (Redner 1982, 1983), and many other models; a review of some of these methods is given in Stanley et al (1982).

3. Current Puzzles

(a) Cluster Structure Near p_c

Although there has been great progress in understanding the critical phenomena of percolation, there is one important question that has not been adequately answered. Namely, "What is the qualitative picture of the infinite cluster just above p_c?" The answer to this question would provide intuition for understanding the properties of transport in random media. In the recent past, two models of cluster structure have been introduced. One is the de Gennes-Skal-Shklovskii model (1975, 1976), in which the percolating cluster is viewed as an quasi-regular lattice of nodes -- points where there are at least three independent paths leading to infinity (Fig. 7(a)). The characteristic spacing between the nodes is assumed to be proportional to ξ . These considerations lead to a homogeneous lattice which is scale similar to a regular lattice if all lengths are rescaled by a factor of ξ . It is then possible to calculate transport properties in terms of series and parallel combinations of the "macrolinks" which join neighbouring nodes. An ingredient which is still lacking in this picture is a model for these macrolinks.

A more accurate picture of cluster structure would incorporate the fact that it is self-similar for length scales $< \xi$. The "fractal" model attempts to account for this self-similar aspect by representing the infinite cluster as a regular self-similar fractal object on which problems may be solved exactly (Gefen et al 1981). The Sierpinski gasket, shown in Fig. 7(b), is a typical fractal that is obtained by removing a central triangle of linear dimension $\ell/2$ from a solid triangle of linear dimension ℓ . The construction is repeated indefinitely in all of the smaller solid triangles produced at each stage of construction. The resulting structure possesses no characteristic scale and has a non-integral Hausdorff or fractal dimensionality. Upon increasing the magnification by a factor of 2, the area of the fractal increases by a factor of 3 (rather than a factor of 4 for a homogeneous two-dimensional object) leading to a <u>fractal dimension</u> $\bar{d}$ = log 3/log 2.

The de Gennes-Skal-Shklovskii model works well in high dimension where the formation of closed loops on many length scales does not dominate. In lower dimensions, loop formation becomes relatively more important and the fractal model works better. However, neither model is sufficient to give a complete description of cluster structure. The essential feature that has yet to be incorporated into a specific model is a self-similar structure for length scales less than ξ and a homogeneous structure for scales larger than ξ . An attempt to represent this situation is illustrated in Fig. 7(c), by incorporating elements of both Figs. 7(a) and 7(b).

The resulting "nodes, links and blobs" model has been proposed by Coniglio (1981a,b, 1982), Pike and Stanley (1981), and Stanley (1981). In this picture, it is possible to divide cluster bonds into three classes: (i) "dangling ends" that do not contribute to transport, (ii) "multiply-connected" bonds in the self-similar blobs, and (iii) "cutting" bonds, which if cut, cause two previously chosen connected sites to become disconnected. Very interesting quantitative information can be obtained for the mean number of bonds in each of the three classes. Moreover, the cutting bonds play a dominant role in transport properties, and explicit predictions for experiments on inhomogeneous systems can be made.

Although these recent advances are quite promising, the basic issue of a simple and quantitatively correct picture for the percolating cluster is still unresolved.

(b) The conductivity problem.

An important variation of percolation is the random resistor network, obtained if each bond in a percolating sample is assigned a fixed resistance. Such an idealized model has been used to describe the conductivity of random metal-insulator composites (cf. Table 1). A basic feature of the random resistor network is that the conductivity vanishes as a power law

$$(p - p_c)^t, \tag{8}$$

as $p \downarrow p_c$, thereby defining the conductivity exponent t. The calculation of this exponent is of fundamental theoretical significance in attempting to relate cluster structure and transport. Most of the theoretical attention has been confined to two-dimensional networks because of the computational difficulties associated with higher-dimensional lattices. Early work gave a wide spread in values for the conductivity exponent, but more recent estimates for t appeared to converge to a value of approximately 1.3 (Lobb and Frank 1979, Fogelholm 1980). It was thought that the coincidence of the value of t with $\nu = 4/3$ was not accidental, and that this fact might provide a bridge between cluster structure and transport. In the past year, however, two very accurate studies based on finite-size scaling gave $t = 1.28 \pm 0.02$ (Derrida and Vannimenus 1982), and $t = 1.30 \pm 0.02$ (Frank and Lobb 1983), apparently excluding the conjecture $t = \nu$ in two dimensions.

In attempting to resolve this situation, Alexander and Orbach (1982) made an important conceptual advance by studying a variety of physical problems on self-similar structures such as fractals. From scaling considerations, they were led to define a new fundamental dimension, the <u>fracton dimension</u> $\tilde{d}$, associated with the scaling behavior of the density of states on fractals, distinct from the fractal dimension of the structure. Roughly speaking, the fracton dimension may be thought of as a fractal dimension in reciprocal space. Each eigenmode of a linear problem, such as linear oscillations or diffusion on a fractal is represented by a point in reciprocal space. The density of these points near the origin scales anomalously, leading to new predictions concerning the frequency dependence of the density of states (Rammal and Toulouse 1983). From existing numerical data, it appears that the fracton dimension of percolating clusters at p_c is 4/3, independent of the embedding spatial dimension. Assuming this result to be exact, and employing the connection between diffusion and conductivity, Alexander and Orbach conjectured that $t = 91/72 \cong 1.264$ in two dimensions, in reasonable agreement with the latest numerical work. It will be interesting to see whether their predictions will be ultimately borne out in two dimensions, and in three dimensions where there is presently a lack of accurate numerical data.

Since these new predictions are intimately connected with understanding diffusion on self-similar objects, it is worthwhile to consider this problem in some detail.

(c) Novel physics on self-similar structures: The diffusion problem.

Consider the problem of a discrete random walker on a percolating cluster (whimsically termed the "ant in the labyrinth " by de Gennes). If $p = 1$, then the medium is homogeneous and the mean-square displacement varies with time as,

$$\langle r^2 \rangle \simeq Dt, \qquad t \rightarrow \infty \quad \text{or} \quad r \rightarrow \infty. \tag{9}$$

As the medium becomes random, the diffusion coefficient D decreases, and vanishes as $p \downarrow p_c$. It is most convenient to write this critical behavior as a function of for what follows. Thus as $p \downarrow p_c$, ξ diverges and D vanishes as

$$D \simeq \xi^{-\theta}. \tag{10}$$

From the Einstein relation between the conductivity and the diffusion coefficient, the exponent $\theta = (t - \beta)/\nu$.

It is important to emphasize that D has the behavior indicated in Eq. (10), only if the diffusing particle has travelled a distance much greater than ξ. In this limit, the medium appears homogeneous and Fick's law, Eq. (9) applies. To obtain the behavior in the limit $r < \xi$, we use an elegant scaling argument of Gefen et al (1983).

The diffusion coefficient should be thought of as a function of the length scale r over which the diffusion takes place. Since r and ξ are the only two parameters in the problem, and since the behavior of D(r) for $r \gg \xi$ is given by Eq. (10), we write

$$D(r) \simeq \xi^{-\theta} f(r/\xi). \tag{11}$$

In the limit $r \ll \xi$, the scaling function f is assumed to be a simple power law of its argument. Additionally, in this short time regime, ξ must drop out of the problem; the diffusing particle has not yet detected what the value of ξ should be. These considerations lead to

$$D(r) \simeq \xi^{-\theta}(r/\xi)^{x}$$

$$\Longrightarrow r^{-\theta} \quad \text{for} \quad r \ll \xi. \tag{12}$$

As a result, we find anomalous diffusion in the short time regime

$$(d/dt)\ \langle r^2 \rangle \simeq D(r) \simeq r^{-\theta}$$

$$\Longrightarrow \langle r^2 \rangle \simeq t^{(2/2+\theta)}. \tag{13a}$$

An alternative form of the latter equation is,

$$\langle r^2 \rangle^{\bar{d}} \simeq t^{\tilde{d}} \tag{13b}$$

This indicates that Fick's law, $\langle r^2 \rangle \simeq t$, is fundamentally modified on a fractal. Anomalous length rescaling properties enter through the fractal dimension $\bar{d}$, while anomalous time or frequency rescaling properties appear through the fracton dimension $\tilde{d}$ (Rammal and Toulouse 1983).

The transition from the short-time anomalous behaviour to the asymptotic Fick's law behavior is shown in Fig. 8. The scaling argument predicts a crossover between the self-similar regime for small distance scales, to the homogeneous regime in the opposite limit at a "crossover" time, t_x, which scales as $\xi^{2+\theta}$. This is the time needed for a particle to diffuse a distance of the order of ξ. Presently, analytical and numerical work is in progress to check the predictions of anomalous diffusion.

4. Concluding remarks.

In summary, percolation is a very simple and appealing model for describing the physical properties of random media. Although extensive studies have provided much information about the model, there are still several open questions at the basic qualitative level. Furthermore, a simple picture of cluster structure near the percolation threshold is still lacking. As a consequence, it is still not clear if there exist direct relationships between cluster structure and transport in random media. Finally, a rapidly emerging area is the study of simple physical problems on self-similar structures. On these structures, translational symmetry is lost, but dilation symmetry holds. This feature appears to be the source of many new and intriguing physical effects which are currently under intensive study.

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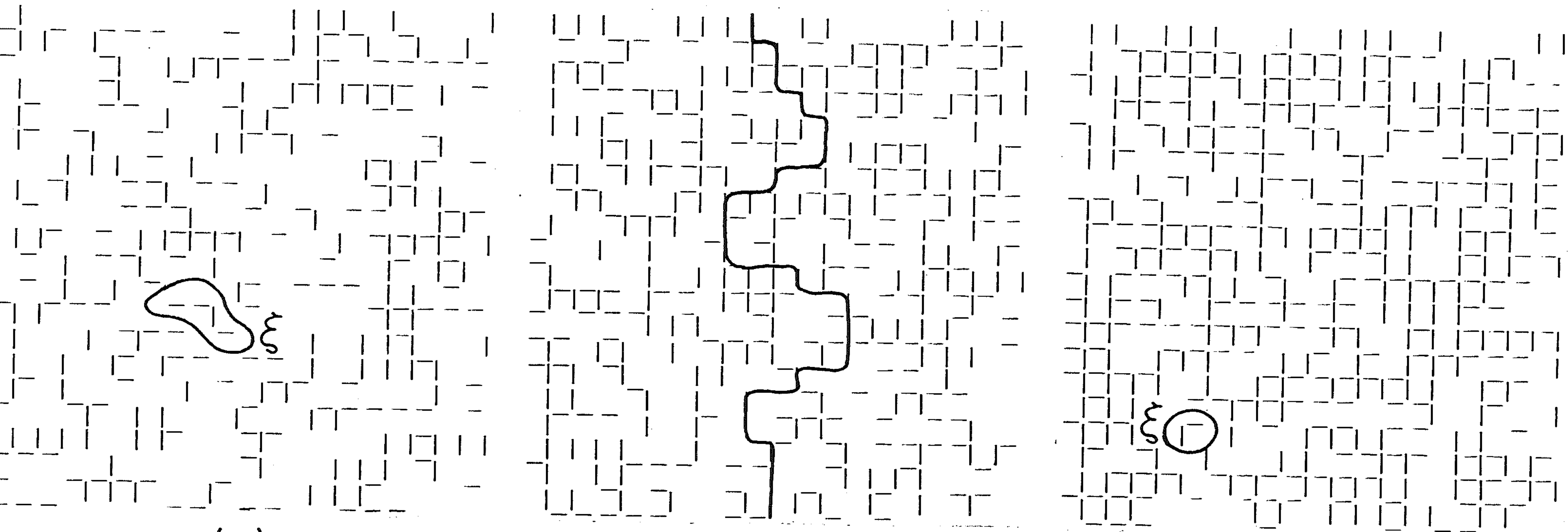


(a) (b) (c)

Fig. 1

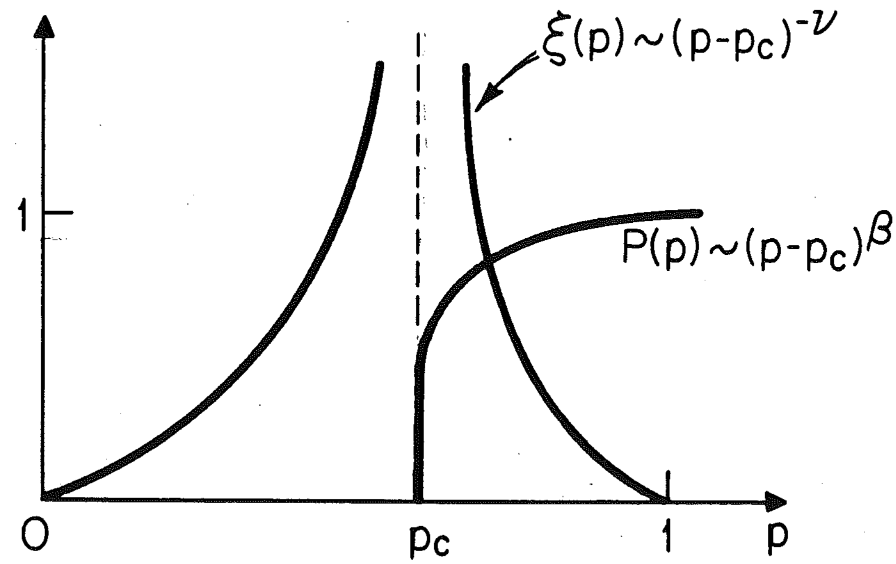
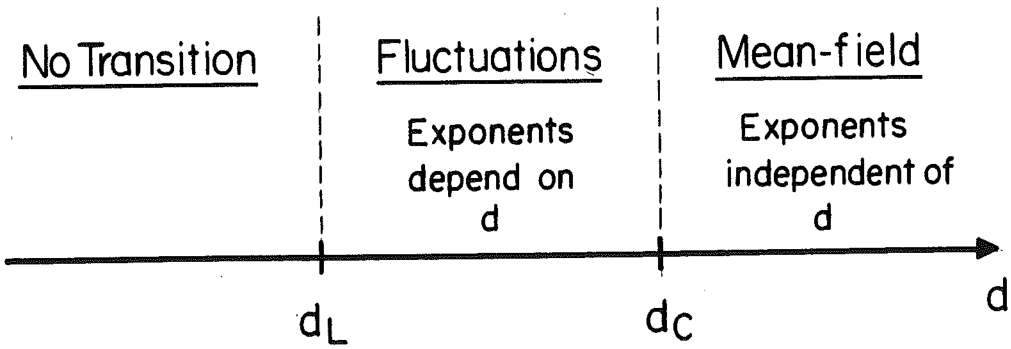


Fig. 2



No Transition
Fluctuations
Exponents depend on d
Mean-field
Exponents independent of d
d_L
d_C
d

Fig 3

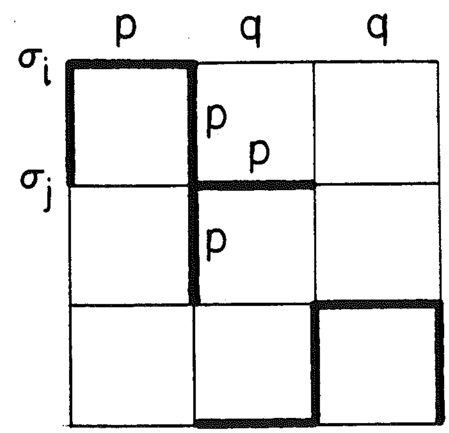
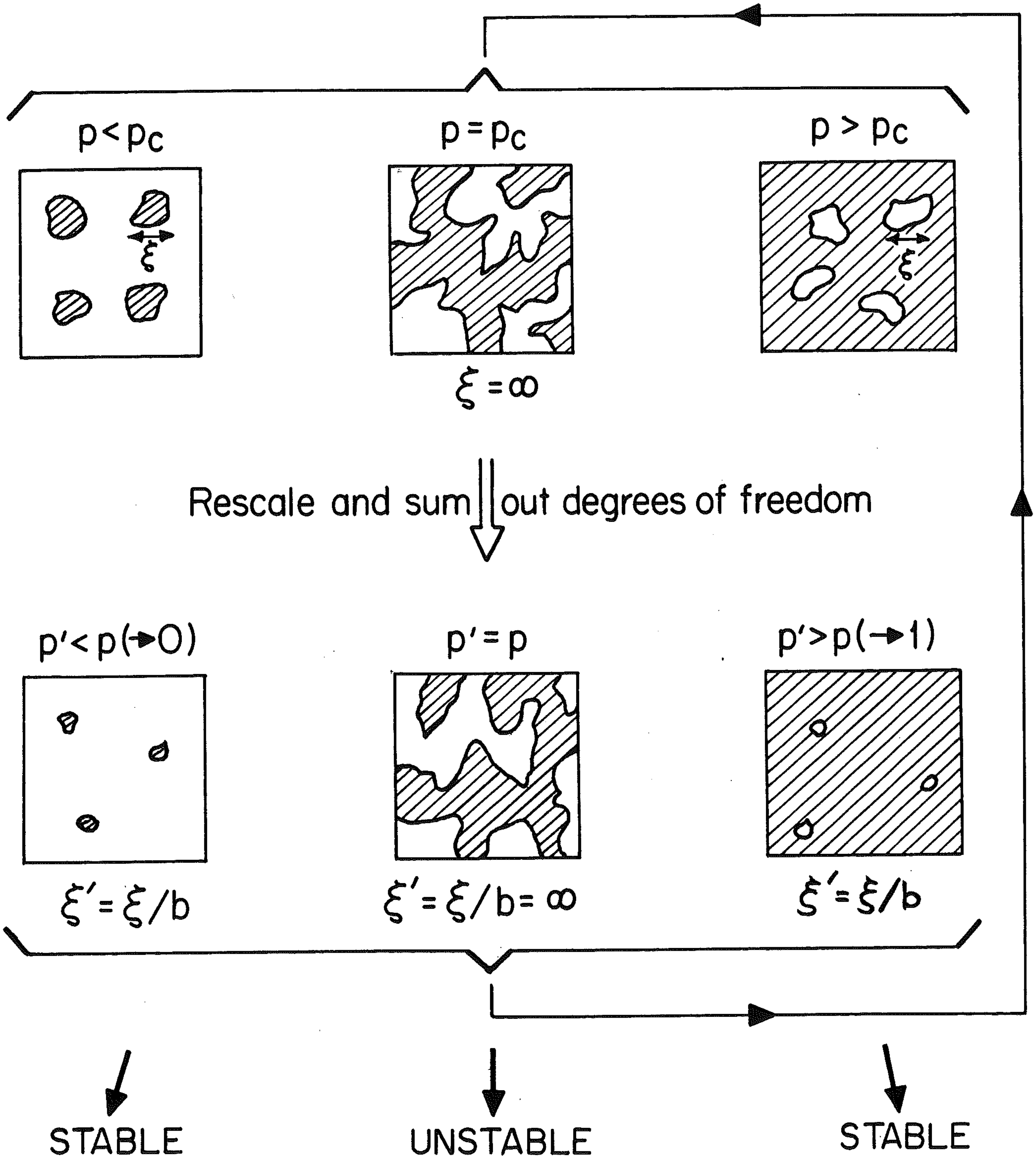


Fig. 4



p<pc
p=pc
p>pc
ξ
ξ=∞
ξ
Rescale and sum out degrees of freedom
p'<p(→0)
p'=p
p'>p(→1)
ξ'=ξ/b
ξ'=ξ/b=∞
ξ'=ξ/b
STABLE
UNSTABLE
STABLE

Fig 5

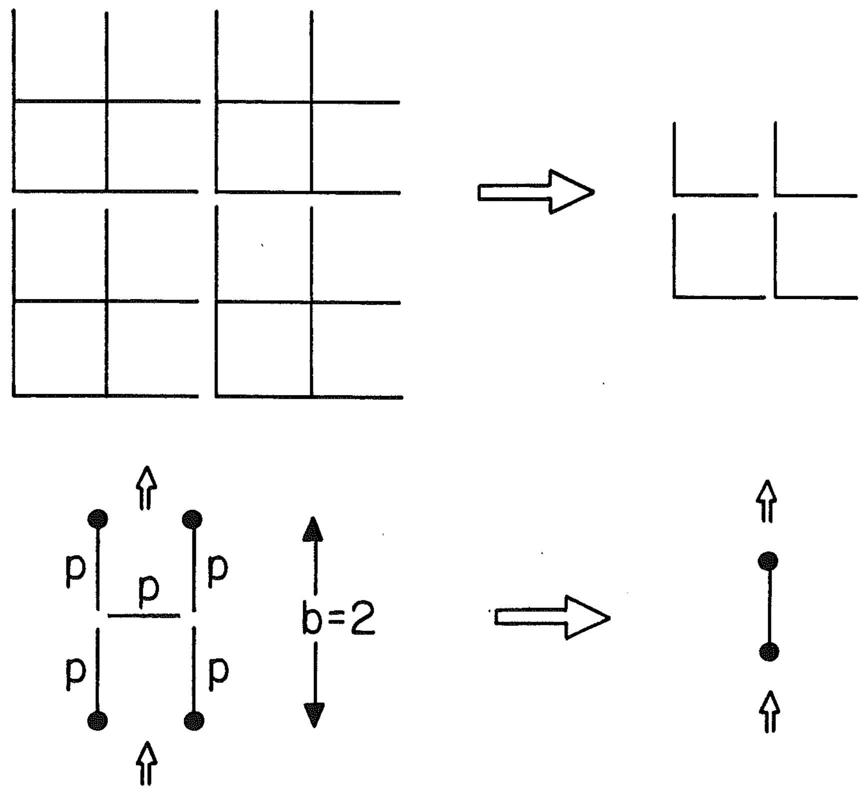


Fig. 6

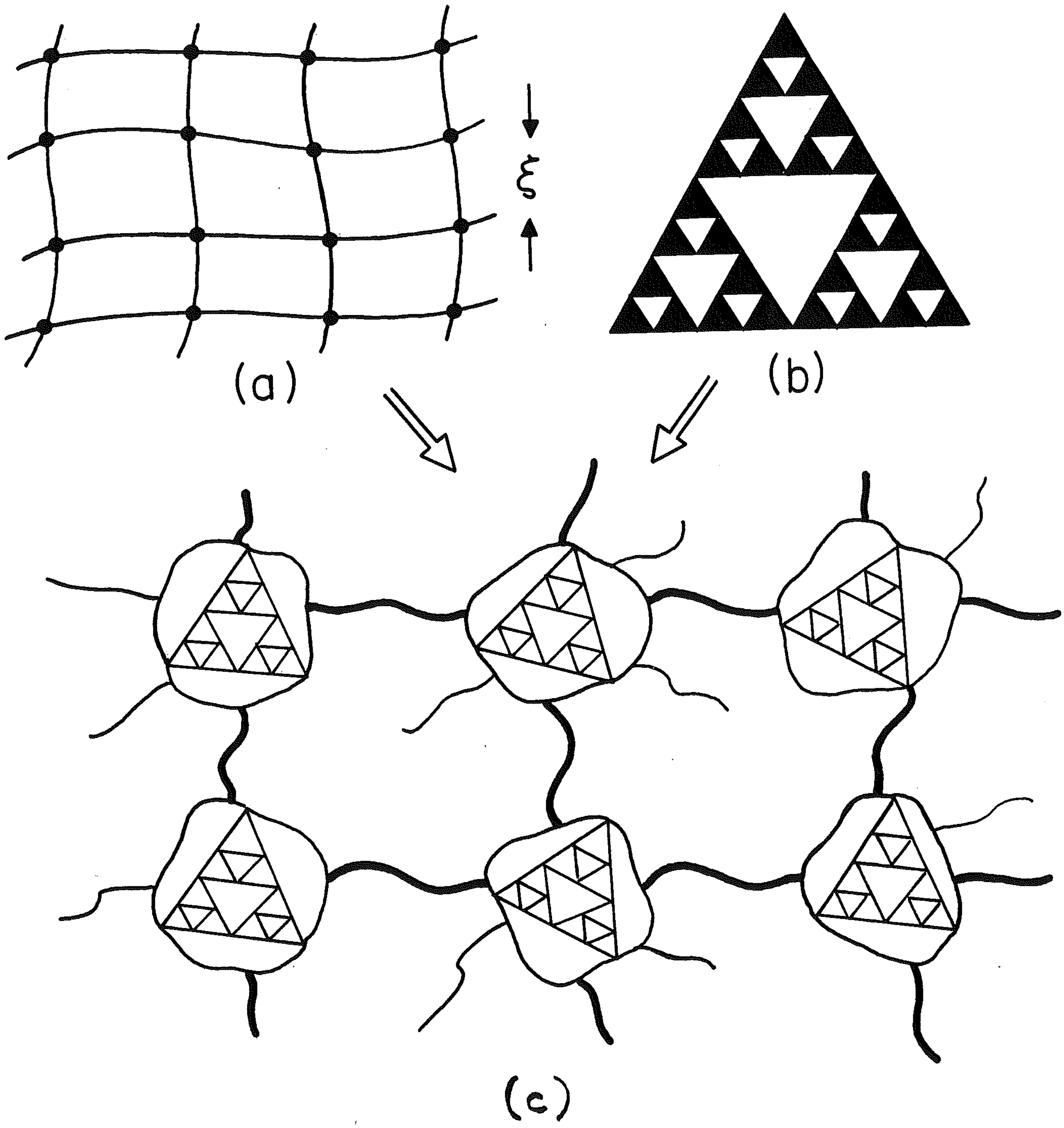


Fig 7

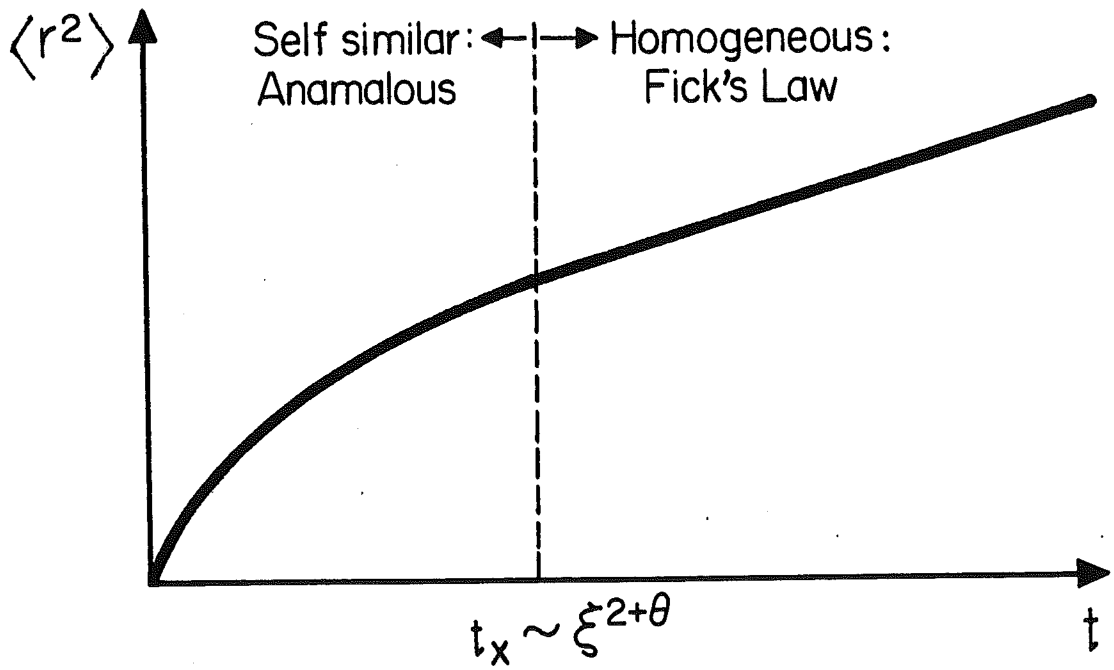

⟨r²⟩
Self similar:
Anamalous
Homogeneous:
Fick's Law
$t_x \sim \xi^{2+\theta}$
t

Fig 8