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LETTER TO THE EDITOR

Exact exponent relations for random resistor–diode networks

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Abstract. We investigate the percolation properties of a random network of non-directed bonds (resistors) and *arbitrarily* oriented directed bonds (diodes). For the square lattice, there is a multicritical line which connects the isotropic percolation threshold with a network fully occupied by randomly oriented diodes ('random Manhattan'). Along this line, symmetry and invariance properties are used to demonstrate that the correlation length exponents are constant. Furthermore, a lattice independent relation is proved which shows that at the isotropic percolation threshold, the correlation length diverges with the same exponent when the transition is approached by varying either the resistor concentration or the concentration of randomly oriented diodes.

Recently, there has been considerable attention devoted to directed percolation, a geometrical model in which directed bonds of a fixed orientation connect lattice sites (for a current review see e.g. Kinzel (1982)). Much of the interest in this model stems from its novel anisotropic critical behaviour. Applications of directed percolation include Reggeon field theory (Cardy and Sugar 1980) and branching Markov processes (Schlögl 1972) which arise in chemical reaction models.

In this letter, we consider a more general percolation model containing directed bonds (diodes) of *arbitrary* orientation as well as non-directed bonds (resistors). Our motivation for this study arises in part because this model has much richer geometrical behaviour than either directed or pure isotropic percolation since connected paths mediated by an arbitrary combination of resistors and diodes can form (Redner 1981, 1982). In addition, this resistor–diode percolation model may be relevant for describing various aspects of information theory and communication network problems (see e.g. Ford and Fulkerson 1962, Frank and Frisch 1971). The primary results of this work are to derive general relations for the correlation length exponents of the network. In particular, for the square lattice we shall demonstrate that correlation length exponents are constant along a multicritical line in the phase diagram. In addition, at the isotropic percolation threshold, the correlation length diverges with the same exponent if the transition is approached by varying either the resistor concentration or the concentration of randomly oriented diodes. This second result is valid for all lattices. Both exponent relations are obtained within the framework of the position-space renormalisation group (PSRG) with an arbitrary rescaling factor. Since the PSRG is expected to become exact in the infinite cell-size limit (Reynolds *et al* 1980), our results for the exponents should become exact as well.

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On the square lattice, the random resistor–diode network is defined by joining nearest-neighbour sites by a positive diode (conducting either upward or to the right), a negative diode (conducting in the opposite direction), a resistor, or the sites may be disconnected with respective probabilities p_+, p_-, p and $q = 1 - p_+ - p_- - p$ (figure 1(a)). From a previous small-cell PSRG calculation, an extremely symmetric phase diagram was obtained (figure 1(b)). Our interest is primarily on the multicritical line where four phases in the system are simultaneously critical (figure 1(c)). This line is defined by both reflection symmetry ($p_+ = p_-$) and dual symmetry ($p = q$). One end

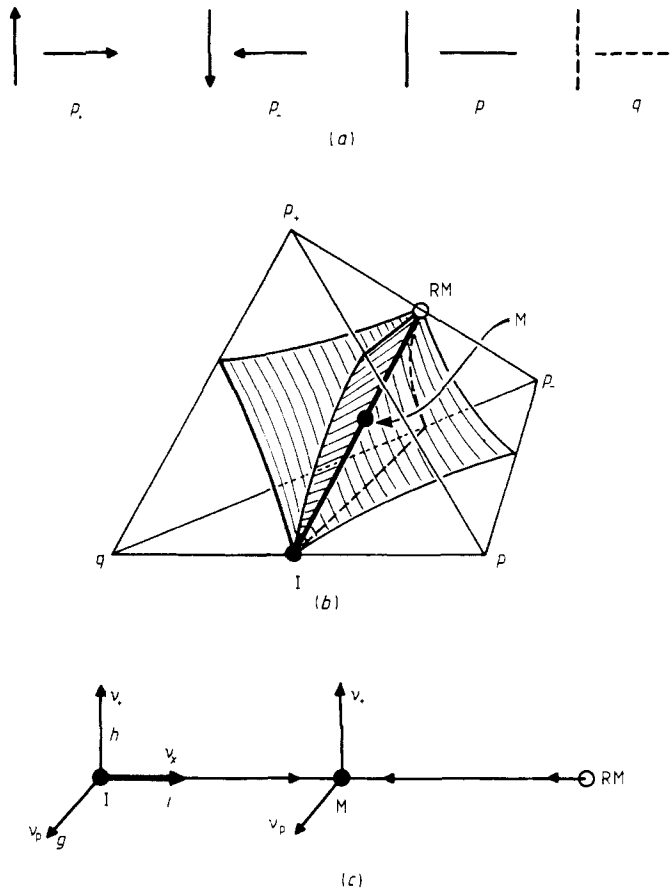

Figure 1. (a) Definition of the bond elements which comprise the random resistor–diode network on the square lattice. Isotropic percolation is recovered if $p_+ = p_- = 0$, while directed percolation is the special case $p = 0$, and one of p_+ or $p_- = 0$. (b) Composition tetrahedron for the network. For any point in the figure, the perpendicular distance to any face gives the relative concentration of the bond species labelled at the opposite apex. Second-order surfaces (shaded) divide the diagram into four phases: positive diode, negative diode, resistor and non-conducting. They meet along a common multicritical line (bold line) which is reflection and dual symmetric. This figure has appeared previously and is included for completeness. (c) The multicritical line from (b). At the isotropic

describes the isotropic percolation point (I) at $p = q = \frac{1}{2}$, while the other end describes a network completely filled with randomly oriented diodes ($p_+ = p_- = \frac{1}{2}$). It has the structure of a 'random Manhattan' (RM). The PSRG flow emanates from these two points and meets at a 'mixed' fixed point (M) which turns out to be very close to the midpoint of the line.

At the points I and M, two distinct correlation lengths diverge. One divergence is observed by varying $p - q$ with $p_+ - p_-$ fixed to be zero, so that reflection symmetry is preserved. This causes a transition between the non-percolating and resistor phases, and the isotropic correlation length diverges as $(p - p_c)^{-\nu_p}$, where $p_c = \frac{1}{2} - p_+$, and ν_p is the correlation length exponent of isotropic percolation whose numerical value is $\frac{4}{3}$ (den Nijs 1979). The second length diverges by varying $p_+ - p_-$, with $p - q$ fixed to be zero so that dual symmetry is preserved. This generates a transition between diode phases of opposite orientation. As the diode 'polarisation' approaches zero, the correlation length 180° opposite to the polarisation diverges as $p_\pm^{-\nu_+}$.

A striking result from the small-cell PSRG is that ν_p and ν_+ are constant along the multicritical line. In addition, a third exponent at I, ν_x, describing the divergence of the isotropic correlation length as the fixed point is approached along the multicritical line, equals ν_p (figure 1(*c*)). We shall demonstrate that these results are true for a PSRG with an arbitrary rescaling factor.

The first result can be derived using the observation (J L Cardy, private communication; see also Arrowsmith and Essam (1977)) that the probability, $P(i \to j)$, that site i connects to site j (independent of whether there is a connected path from j to i) is a function only of the variables $A_\pm \equiv p_\pm + p$. On the multicritical line, $A_+ = A_- = \frac{1}{2}$, hence $P(i \to j)$ is a constant. In addition, $P(i \to j)$ is equal to a linear combination of PSRG recursion relations. If p' and $p'_\pm$ are the respective probabilities that a cell can be traversed in both directions, and in only one direction, then $P(i \to j) = p'_\pm + p'$, if i and j refer to opposite edges of the cell.

This equivalence can be applied most effectively if the recursion relations are rewritten in terms of the fields

$$g \equiv p - q = A_+ + A_- - 1, \qquad h \equiv p_+ - p_- = (A_+ - A_-)/2,$$
$$i \equiv (p_+ + p_-) - (p + q) = 2(p_+ + p_-) - 1 = 1 - 2(p + q), \tag{1}$$

whose geometric interpretation is indicated in figure 1(*c*). The first two quantities are merely the dual and reflection symmetry breaking fields respectively (Dhar *et al* 1981, Redner 1982), while the third quantity measures the directedness of the system. For example, $i = +1$ at RM, and $i = -1$ at I. In terms of these coordinates, all mixed derivatives of the form $\partial\alpha'/\partial\beta$, $(\alpha, \beta = g, h, i)$ with $\alpha \neq \beta$ vanish by either dual or reflection symmetry. On the other hand, $\partial g'/\partial g$ and $\partial h'/\partial h$ depend on A_+ and A_- only, so that they are constant on the multicritical line. The derivative $\partial i'/\partial i$ is non-constant, so that the linearised transformation matrix $T_{\alpha\beta} \equiv \partial\alpha'/\partial\beta$ is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{2}$$

with $a = \partial g'/\partial g = L^{1/\nu_p}$, $b = \partial h'/\partial h = L^{1/\nu_+}$, $c = \partial i'/\partial i = L^{1/\nu_x}$ and L the rescaling factor in the PSRG transformation.

Thus the T-matrix is diagonal, and more importantly, the exponents ν_p and ν_+ are *constant* along the multicritical line. As a consequence, the critical behaviour

along this line can be quantified. For example, at RM, if the transition is approached by varying the diode polarisation, the correlation length opposite to the polarisation diverges with an exponent ν_+. On the other hand, if the transition is approached by occupying the lattice with randomly oriented diodes until a completely filled random Manhattan is attained, then the isotropic correlation length diverges with the exponent ν_p.

To show the second result that $\nu_p = \nu_x$ at I, first consider the eigenvalue a, which by the chain rule equals

$$\partial p'/\partial p + (\partial p'_+/\partial p + \partial p'_-/\partial p)|_{h,i}. \tag{3}$$

The quantities $p'_\pm$, and hence $\partial p'_\pm/\partial p$, must contain at least one power of $p_\pm$. At I, these derivatives are equal to zero because $p_\pm = 0$. The remaining derivative $\partial p'/\partial p$ equals the average number of cutting bonds (Coniglio 1981, 1982, Pike and Stanley 1981) in all configurations which span a cell (figure 2). (A cutting bond is defined as one which, if cut, renders the configuration disconnected (Stanley 1977).)

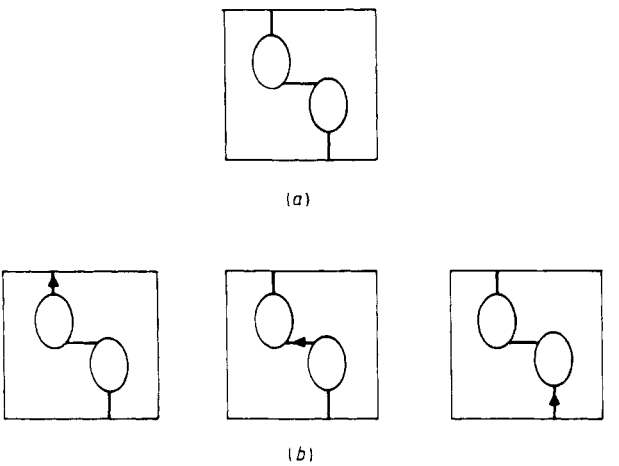

Figure 2. Pictorial representation of the eigenvalues $a = \partial g'/\partial g$ and $c = \partial i'/\partial i$. In ($a$) we show a typical isotropic traversing configuration containing 'blobs' of multiply connected bonds and three cutting bonds. The effect of taking $\partial g'/\partial g$ is to count these cutting bonds. (b) shows closely related configurations in which one of the three cutting bonds is directed. The effect of taking $\partial i'/\partial i$ is to count only the directed cutting bonds. This is equal to the total number of isotropic cutting bonds in (a).

Now consider $c = \partial i'/\partial i$, which by the chain rule is equal to

$$\partial p'_+/\partial p_+ + \partial p'_+/\partial p_- - \partial p'_+/\partial p|_{g,h}. \tag{4}$$

Clearly, the third term is zero at I, and since we will set $p_\pm = 0$ after differentiation, it suffices to consider p'_+ only to first order in p_+ and p_-. For $\partial p'_+/\partial p_+$ to be non-zero, there must be only one p_+ bond in the configurations contributing to p'_+. Since these configurations traverse in only one direction, the single diode *must* be a cutting bond (figure 2). Thus the derivative counts the average number of positive diode cutting bonds in configurations which are isotropic except for the single diode. The sum, $\partial p'_+/\partial p_+ + \partial p'_+/\partial p_-$, counts the number of cutting bonds of either orientation which

force an otherwise isotropic configuration to percolate in only one direction. This number is identical to the total number of isotropic cutting bonds, thus demonstrating that the eigenvalues a and c are equal at I, or equivalently, $\nu_p = \nu_x$. This result holds for all lattices, since the relations for the cutting bonds are lattice independent. Due to this equality, the linearised PSRG transformation at I is isotropic within the reflection symmetry plane. Thus if I is approached for *any* direction in this plane, the correlation length will diverge with the same exponent ν_p.

Further information about the critical behaviour of the network can be obtained by examining the properties of the recursion relations near RM. This is not a fixed point as rescaling random Manhattan configurations leads to both isotropic percolating and non-percolating configurations. Thus the renormalisation flow points away from RM, so that a fixed point between I and RM should exist in general which controls the critical behaviour along the entire multicritical line. On this line, there are two distinct fixed points, I and M, which have the *same* numerical values for the correlation length exponents. At both points, the four phases in the system are simultaneously critical, but the nature of the criticality is qualitatively different. At I, the critical point fluctuations involve only isotropic percolating and non-percolating configurations. On the other hand, at M, there is a non-zero concentration of all four bond species. This suggests that critical fluctuations should include the positive and negative diode phases as well as the resistor and non-percolating phases. This novel fluctuating behaviour should occur all along the multicritical line as M controls the critical behaviour on this line.

In summary, we have studied the percolation properties of a random network of resistors and arbitrarily oriented diodes. For the square lattice, there is a multicritical line in the phase diagram along which correlation length exponents can be related to the exponents of the pure percolation problem. In addition, a lattice independent argument indicates that at the isotropic percolation threshold, the correlation length diverges with the same exponent when the transition is approached from any direction in the phase diagram as long as the diode polarisation is zero. These two results are exact if the PSRG is exact in the infinite cell-size limit. From these exponent relations, we have deduced the critical behaviour of the network on the multicritical line. This behaviour is controlled by a 'mixed' fixed point where critical fluctuations involve the four phases of the system. Many of these predictions for the exponent values may be tested by numerical simulations.

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