ON THE CROSSOVER EXPONENT FOR ANISOTROPIC BOND PERCOLATION

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We show that the exponent which describes crossover between d- and $(d-1)$-dimensional percolation in an anisotropic system equals the mean-size exponent of the $(d-$dimensional system. This situation is analogous to crossover behaviour in anisotropic thermal critical phenomena.

Consider bond percolation on a d-dimensional hypercubical lattice in which the occupation probability for bonds lying within $(d-1)$-dimensional layers perpendicular to z is $p_\perp$, while the occupation probability for bonds parallel to z is $p_z \equiv Rp_\perp$. This anisotropic model was first introduced for $d = 2$ [1,2] and later studied in detail for all d [3]. Very recently this problem has also been treated by position-space renormalization group [4–7] for $d = 2$.

A primary focus in studying this anisotropic model is to describe crossover between d- and $(d-1)$-dimensional percolation as $R \to 0$. The quantitative nature of this crossover may be studied by first formulating a scaling hypothesis for the percolation thermodynamic functions. Here we will consider the mean-size function for a d-dimensional system S_d. About the $(d-1)$-dimensional limit, $R \to 0$, we assume that S_d is a generalized homogeneous function of its scaling fields $\Delta p_\perp \equiv p_\perp - p_{\perp 0}$ and R [3],

$$\lambda^{a_S} S_d(\lambda^{a_{\Delta p_\perp}} \Delta p_\perp, \lambda^{a_R} R) = S_d(\Delta p_\perp, R) . \quad (1)$$

Here the a's are the scaling powers of the corresponding scaling fields. An important consequence of these scaling relations is that there exists a "constant gap" relation for the divergence of successive derivatives of S_d with respect to R. By choosing $\lambda = (\Delta p_\perp)^{-1/a_{\Delta p_\perp}}$ and performing the derivative we find

$$\partial^n S_d / \partial R^n |_{R=0} \sim (\Delta p_\perp)^{(a_S + n a_R)/a_{\Delta p_\perp}} \sim (\Delta p_\perp)^{\gamma^{(n)}} \quad (2)$$

That is, the divergence of successive derivatives of S_d increases with a constant gap,

$$\gamma^{(n)}_{d,d-1} = \gamma_{d-1} + n\phi_{d,d-1} . \quad (3)$$

Here, γ_{d-1} is the mean-size exponent of the $(d-1)$-dimensional system, and this equation serves to define the crossover exponent $\phi_{d,d-1}$.

In ref. [3], using bond counting [‡1] to define the mean cluster size and series expansion methods, it was proved that $\phi_{d,d-1} = \gamma_{d-1}$ for $d = 2$. However series analysis for $d = 3$ did not support this equality, in contrast to what happens in anisotropic thermal critical phenomena. However, by using site counting, we demonstrate that in fact $\phi_{d,d-1} = \gamma_{d-1}$ for all d. To derive this result, we first show that $\partial S_d/\partial R|_{R=0} = 2p_z S^2_{d-1}$ [8]. We calculate $\partial S_d/\partial R|_{R=0}$ by using the analogue of the fluctuation–dissipation theorem applied to percolation [9]. Using site counting we have

$$S_d(p_\perp, p_z) = \sum_{s'} L(s, s') . \quad (4)$$

Here $L(s, s')$ is the pair-connectedness function which is defined as the probability that sites s and s' are in the same finite cluster. For $\partial S_d/\partial R|_{R=0}$, the only terms in the pair-connectedness which contribute, are linear

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[‡1] In bond percolation, the size of a cluster may be defined as either the number of bonds (bond counting), or the number of sites (site counting) in the cluster.

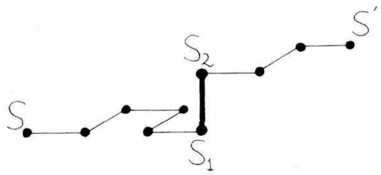

Fig. 1. A typical graphical contribution to $L(s, s')$ which is linear in R. The light bond lie in one $(d-1)$-dimensional layer (here the $x-y$ plane), and the heavy bond is in the z-direction.

in R. A graphical representation of such a term is illustrated in fig. 1. Sites s and s' must lie in adjacent $(d-1)$-dimensional layers, and there is a single z bond joining s and s' via s_1 and s_2.

Thus from the figure, it is clear that we may write for the term linear in R in the pair-connectedness

$$L(s, s') = \sum_{s_1} L(s, s_1) p_z L(s_2, s') . \tag{5}$$

Since the right-hand side of eq. (5) involves the product of pair-connectedness functions within single $(d-1)$-dimensional layers, we obtain after summing over s',

$$\partial S_d / \partial R|_{R=0} = 2 p_z S_{d-1}^2 . \tag{6}$$

The factor 2 arises because site s' may be either one lattice spacing above or below site s. Using this in eq. (3) it follows immediately that $\phi_{d,d-1} = \gamma_{d-1}$.

When bond counting is used, the expression for $\partial S_d / \partial R|_{R=0}$ is no longer easy to derive. However, it is possible to convince oneself that asymptotically, $\partial S_d / \partial R|_{R=0} \sim S_{d-1}^2$ with the constant of proportionality a complicated but non-singular function of p. Thus in bond counting $\phi_{d,d-1} = \gamma_{d-1}$ also. The result (6) can also be obtained indirectly via the anisotropic q-state Potts model, which in the limit $q \to 1$, corresponds to bond percolation with site counting [10].

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