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SUPERDIFFUSION IN RANDOM VELOCITY FIELDS

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Stochastic transport in a medium containing random, but spatially correlated velocity fields is discussed. This type of disorder generally leads to *superdiffusive* behavior in which the mean-square displacement of a random walk, $\langle x^2(t)\rangle$, grows faster than linearly with time. For a two-dimensional layered medium with y-dependent random velocities in the x-direction $u_x(y)$, $\langle x^2(t)\rangle \sim t^{2\nu}$ with $\nu = 3/4$, and with strong sample-to-sample fluctuations. The probability distribution of displacements, averaged over environments, takes a non-Gaussian scaling form at large time, $\langle P(x, t)\rangle \sim t^{-3/4} f(x/t^{3/4})$, where $f(u) \sim \exp(-u^{\delta})$ for $u \gg 1$, with $\delta = 4/3$. For an isotropic two-dimensional medium with $u_x(y)$ and $u_y(x)$ having the same statistical properties, we find $\nu = 2/3$ and $\delta = (1-\nu)^{-1} = 3$. For the layered medium, the moments of the time for a random walker to first reach a distance x in the longitudinal direction increases as $\langle t^n\rangle^{1/n} \sim x^{4/3}$, possibly modified by logarithmic corrections, however. The probability that the walk has not reached a distance x in a time t decreases asymptotically as $e^{-t/\tau}$, with $\tau \sim x^2$, indicating that more than a single time scale is needed to account for first passage properties.

1. Introduction

The motion of a random walk in a random medium is often *subdiffusive*, where the mean-square displacement, $\langle x^2(t)\rangle$, grows more slowly than linearly with time (see, e.g., refs. [1–3] for recent reviews). The basic mechanism for this phenomenon is that as a function of time, a random walk explores regions of progressively higher "resistance" to transport. Consequently the diffusion coefficient depends on the length scale explored, and this ultimately results in subdiffusive motion. In this article, I discuss a simple and general mechanism based on the coupling between diffusion and convection by spatially random, but temporally static velocity fields, that leads to the complementary situation of *superdiffusion*, where $\langle x^2(t)\rangle$ grows *faster* than linearly in time. Various aspects of this work were performed in collaboration with J.-P. Bouchaud, A. Georges, J. Koplik, F. Leyvraz, and A. Provata.

One physical motivation for this class of models arises is describing ground water transport in sedimentary rocks [4]. Consider a two-dimensional stratified medium of homogeneous but distinct layers, so that transport properties vary between strata. When a pressure drop is applied along the strata, the longi-

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tudinal fluid velocity correspondingly varies from layer to layer. In a center-of-mass frame of reference, therefore, the steady velocities in the x-direction are random zero mean functions of the transverse co-ordinate y. We wish to determine the motion of a random walk, which undergoes pure diffusion in the transverse direction, but which is passively carried by this random convection field. Although the longitudinal bias averaged over an infinite number of layers is zero, the average over the finite number of layers that a random walk visits is a fluctuating quantity, which is a decreasing function of the number of layers sampled. This non-vanishing residual bias underlies superdiffusive transport [4–8].

Another important realization of faster-than-diffusive transport arises in turbulence. At a phenomenological level, theoretical descriptions of turbulent transport have been developed which invoke a diffusion coefficient that is an increasing function of length scale [9, 10]. This feature is naturally built in at a microscopic level for the random convection models that will be discussed in this article. A related avenue of research involves determining the relation between the motion of a particle, which is passively carried by a temporally static, but spatially varying divergenceless velocity field, and the statistical properties of the velocity field [11, 12]. In this work, only very simple velocity fields are considered, which permits a detailed investigation of the ensuing motion.

2. Random unidirectional convection

For a two-dimensional continuum system with random velocities in the x-direction which depend only on the y-coordinate, the motion of a passive test particle is accounted for by the Langevin equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x[y(t)]\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \eta(t)\,, \tag{1}$$

in which the random walk is driven by thermal noise along y ($\langle \eta(t)\eta(t')\rangle = 2D_\perp \delta(t-t')$, where $D_\perp$ is the transverse diffusion coefficient) and by the quenched random convention field $u_x(y)$ along x. (The effects of thermal noise in the x-direction are subdominant with respect to the random convention and are neglected here.) For simplicity, we take the convection field to be a Gaussian white noise in space, $\langle u_x(y)u_x(y')\rangle_c = \sigma\delta(y-y')$, where $\langle\cdots\rangle_c$ denotes an average over all velocity configurations of the medium.

A very simple argument can be given to determine the time dependence of the longitudinal mean-square displacement. We first estimate the effective longitudinal bias at time t within the $\sqrt{D_\perp t}$ layers that a typical random walk visits. Since $u_x(y)$ is a random zero mean function, the average bias within the $\sqrt{D_\perp t}$ layers is

$$\langle u_x \rangle_t = \frac{1}{\sqrt{D_\perp t}} \sum_{y}^{\sqrt{D_\perp t}} u_x(y) \sim \sigma^{1/2}(D_\perp t)^{-1/4} . \tag{2}$$

Correspondingly, the RMS longitudinal displacement at time t, $x_{\mathrm{RMS}}(t) = \langle x^2(t) \rangle^{1/2}$, may be estimated as

$$x_{\mathrm{RMS}}(t) \sim \langle u_x \rangle_t t \sim \sigma^{1/2} D_\perp^{-1/4} t^{3/4} , \tag{3}$$

a remarkable result which was apparently first derived by Matheron and de Marsily [4]. Thus the random walk spreads out at a rate which is *faster* than pure diffusion.

In addition to the superdiffusive transport exhibited by this layered model, there is a lack of self averaging [8, 13]. That is, the asymptotic rate at which the probability distribution spreads in a single environment is different from the spread rate when an average over all environments is taken. This feature can be seen in computing the higher moments of the longitudinal displacement. The moment of arbitrary order can be written formally as

$$\langle \langle x^n(t) \rangle_{\mathrm{w}} \rangle_{\mathrm{c}} = n! \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, \langle \langle u(y(t_1)) \cdots u(y(t_n)) \rangle_{\mathrm{w}} \rangle_{\mathrm{c}} . \tag{4}$$

The double angle brackets indicate that one should first average over all transverse Brownian trajectories for a given configuration of random velocities, and then average over all configurations. However, these two averages factorize and can be performed in either order. Thus the velocity correlation function can be written as

$$\langle \langle u(y(t_1)) \cdots u(y(t_n)) \rangle_{\mathrm{w}} \rangle_{\mathrm{c}} = \int_{-\infty}^{+\infty} \mathrm{d}y_1 \, \mathrm{d}y_2 \cdots \mathrm{d}y_n \, \langle u(y_1) \cdots u(y_n) \rangle_{\mathrm{c}}$$

$$\times p(y_n, t_n) \, p(y_{n-1} - y_n, t_{n-1} - t_n) \cdots p(y_1 - y_2, t_1 - t_2) , \tag{5}$$

where $p(x, t) = (1/\sqrt{4\pi D_\perp t}) \exp(-x^2/4D_\perp t)$ is the Gaussian probability distribution for the transverse motion. The product of Gaussians in eq. (5) is the probability that a Brownian path visits the sequence of transverse positions $\{y(t_i)\}$ at times $\{t_i\}$, having started at $y = 0$. For the continuous model defined by eq. (1), $\langle u(y_1) \cdots u(y_n) \rangle_{\mathrm{c}}$ is a sum of products of delta functions.

Consequently, the second moment is [3, 8]

$$\langle\langle x(t)^2\rangle_{\rm w}\rangle_{\rm c} = 2\sigma \int_0^t {\rm d}t_1 \int_0^{t_1} {\rm d}t_2 \int_{-\infty}^{+\infty} {\rm d}y\; p(0, t_1 - t_2)\, p(y, t_2) = \frac{4\sigma}{3\sqrt{\pi D_\perp}}\, t^{3/2} . \tag{6}$$

On the other hand, the longitudinal displacement, averaged over all walks in a *fixed* environment, $\langle x(t)\rangle_{\rm w}$, depends on the configuration, and does not necessarily converge to zero at large times. However, the average over all environments, $\langle\langle x(t)\rangle_{\rm w}\rangle_{\rm c}$ does equal zero in the center-of-mass reference frame. Clearly $\langle x(t)\rangle_{\rm w}$ has a distribution over environments which is a Gaussian of variance $\langle\langle x(t)\rangle_{\rm w}$ has a distribution over environments which is a Gaussian of variance $\langle\langle x(t)\rangle_{\rm w}^2\rangle_{\rm c}$. This dispersion can also be calculated for the continuous model of eq. (1),

$$\langle\langle x(t)\rangle_{\rm w}^2\rangle_{\rm c} = \sigma \int {\rm d}y\, \langle \mathcal{N}(y, t)\rangle_{\rm w}^2 = (\sqrt{2} - 1)\, \frac{4\sigma}{3\sqrt{\pi D_\perp}}\, t^{3/2} , \tag{7}$$

where $\mathcal{N}(y, t) = \int_0^t {\rm d}t'\, \delta(y - y(t'))$ is the number of times that the random walk $y(t)$ visits layer y after time t, having started at $y = 0$. Thus both the configuration average of the mean-square displacement $\langle\langle x(t)^2\rangle_{\rm w} - \langle x(t)\rangle_{\rm w}^2\rangle_{\rm c}$ and the second moment $\langle\langle x(t)^2\rangle_{\rm w}\rangle_{\rm c}$ vary as $t^{3/2}$, but *with different prefactors* [8, 13]. This implies that there are non-vanishing sample specific fluctuations in the variance, asymptotically, and hence in the probability distribution itself.

Underlying these moments is the probability distribution of displacements, which is the basic characteristic feature of the transport process. The configuration average of this distribution is expected to take a well-defined scaling form in the large time limit,

$$\langle P(x, t)\rangle_{\rm c} \to t^{-3/4} f(x/t^{3/4}) , \tag{8}$$

where it is understood that x and t are simultaneously large with $u \equiv x/t^{3/4}$ finite (as is the case of any central limit theorem). For $u \gg 1$, the scaling function is expected to vary as

$$f(u) \sim \exp(-cu^{\delta}) , \tag{9}$$

where possible power-law prefactors have not been written. Based on our numerical data and on theoretical arguments, we predict that the shape exponent δ has the anomalous value $\delta = 4/3$.

Our numerical approach to test the validity of eq. (9), and to find δ, is based on computing the dimensionless moment ratios [8]

$$n_{2k}(t) \equiv \frac{\langle x^{2k}(t)\rangle}{\langle x^{2(k-1)}(t)\rangle\langle x^{2}(t)\rangle} . \tag{10}$$

If $\langle x^{2k}(t)\rangle \sim \langle x^2(t)\rangle^k$, n_{2k} will approach a constant as $t\to\infty$, whose value depends on k and on δ. By attempting to match our set of numerical estimates for n_{2k} to the corresponding set of values that arise directly from eq. (9), we can infer a value of δ.

For performing this numerical computation, we employ a simple hopping model on the square lattice, in which each horizontal line is randomly assigned a bias $\pm$. At each lattice site, a random walk moves either in the $+y$ or $-y$ direction with probability $\frac{1}{4}$ (thus fixing the value of $D_{\perp}$), or moves parallel to the bias with probability $\frac{1}{2}$. We use exact enumeration [2] to find the probability distribution of longitudinal displacements exactly for a given environment, and then average over *all* environments in a system of finite width w. This averaging is carried out for the 92 205 distinct velocity configurations (cyclic permutations and reflection symmetry), out of the 2^{23} states on a system of width $w=23$ with periodic transverse boundary conditions. This procedure provides the *exact* configurational average probability distribution for an infinite system up to 22 time steps. The qualitative trend from our data is that the effective value of δ is a decreasing function of the order of the moment being analyzed. Our estimate of n_4 is consistent with a value of δ in the vicinity of 1.7, while our estimate for n_8 is consistent with $\delta \leqslant 1.4$. This latter value is expected to be representative of the tail of the probability distribution function.

The apparent change in the estimate for δ as a function of the order of the moment being analyzed suggests that the tail of the distribution function may be governed by anomalous events [8]. Consider, therefore, the averaged probability of finding a "stretched out" walk, namely, $\langle P(x\sim t, t)\rangle_c$. According to eq. (9), this probability should vary as $\exp(-t^{\delta/4})$. On the other hand, the probability of a walk being stretched out can be bounded from below by the probability that the walk remains transversely confined within a region of unidirectional bias. This confining probability, averaged over all environments, is isomorphic to the survival probability of a one-dimensional random walk in the presence of randomly distributed traps [14], and hence varies as $\exp(-at^{1/3})$. By comparing these two distributions, one concludes that $\delta \leqslant 4/3$. Thus the large distance tail of the probability distribution function is controlled by an anomalously large contribution stemming from stretched out walks. This exponent value of 4/3 can also be deduced by a complementary argument which rests on determining the distribution function for the quantity $\int dy\, \langle \mathcal{N}(y,t)^2\rangle$ [8]. Yet to be settled, however, is the range of $x/t^{3/4}$ over which this anomalous tail behavior actually holds.

3. First-passage characteristics for unidirectional random convection

Having discussed some of the basic phenomena related to the probability distribution of displacements at a fixed time, let us now turn to the complementary, and basic problem of the first-passage characteristics of superdiffusive transport. Consider a system of length $2L$ and width w, with periodic boundary conditions in the transverse direction, and absorbing boundary conditions in the longitudinal direction. A uniform line source of random walkers is introduced at $t=0$, i.e., $P(x,y,t=0)=(1/w)\delta(x)$. We are interested in the mean time until the random walks reach the absorbing boundary, or more fundamentally, the probability that a random walk is absorbed at time t.

One way to estimate the first passage time is by computing perturbatively the first correction to the mean first-passage time due to the bias. For non-zero bias, the mean first-passage time to reach $x=\pm L$ starting at (x,y) obeys the recursion relation

$$t_{x,y}=\tfrac{1}{4}(t_{x+1,y}+t_{x-1,y}+t_{x,y+1}+t_{y,x-1})+\tfrac{1}{4}\epsilon(y)\,(t_{x+1,y}-t_{x-1,y})+1\,, \tag{11}$$

where $\epsilon(y)\leq 1$ is the longitudinal bias at transverse position y, and $t_{|L|,y}=0$. For the case of no bias, $\epsilon(y)=0\ \forall y$, it is well-known and easy to derive that $t^{(0)}_{x,y}=\frac{1}{2}(L^2-x^2)$. Using this fact, it can be shown [15] that the first non-vanishing correction to the first-passage time for a random walk starting at $x=0$, average over all transverse starting points and over all configurations, has the form

$$T_L\equiv\langle t_{0,y}\rangle_{y,c}\approx L^2-\epsilon^2L^3+\cdots. \tag{12}$$

We therefore conclude that for $\epsilon>\epsilon_c\sim L^{-1/2}$ a crossover occurs from purely diffusive first-passage characteristics to a new regime of first-passage behavior.

We next assume that at large distances, the mean first-passage time has the form

$$T_L\sim(L/\epsilon)^{\alpha}\,. \tag{13}$$

That is, we expect the first passage time to be a function of only the ratio of the only two length scales in the problem: the length of the system and the microscopic length imposed by the bias. By matching eq. (13) to the first-passage time for pure diffusion at $\epsilon = \epsilon_c$, we deduce that $\alpha = 4/3$. Thus it appears that the exponent α is simply equal to the inverse of the size exponent ν, as might be expected naively[#1].

An alternative, and simpler approach is based on first calculating the exact first-passage time in a system of length L and width w, in the limit $L/w \to \infty$, and then using a simple crossover picture to infer the behavior in the interesting limit of L, $w \to \infty$, with L/w finite. In the former limit, the random walk uniformly samples the cross-sectional area of the system. Thus for the discrete hopping model introduced in the previous section, the average mean-first passage time is

$$T_L = L\langle 1/v \rangle \,, \tag{14}$$

where v is the bias in a given configuration, and the average is taken over all configurations. Upon performing this average for a binomial distribution of velocities, one finds

$$T_L \propto L\sqrt{w}\ln w \,. \tag{15}$$

This form is expected to be correct for $w^2 \ll T_L$. However, when w^2 becomes of the same order as T_L, the first passage time should no longer depend on w. Thus for all $w^2 > T_L$, one expects that the mean first-passage time will be of the same order as that given by eq. (15) when $w^2 \simeq T_L$. This crossover argument yields

$$T_L \sim (L \ln L)^{4/3} \,. \tag{16}$$

In this discussion, it has been tacitly assumed that w is odd, so that every configuration will have a non-zero bias. For w even, so that some configurations have zero bias, an approach of a similar spirit yields consistent results.

To test this prediction, we have performed numerical calculations for first-passage properties which are based on exact enumeration over all walks and all starting points for a single environment, and then averaging over all environments. To average over all starting points, we consider a uniform line sources of random walks which start at $x = 0$, and which are absorbed upon

[#1] In my oral presentation, I gave what I believe now to be an erroneous invocation of this argument.

reaching $x = \pm L$. First we compute $T(w, L)$, the mean first passage time for a sequence of systems of increasing width w and fixed length L, and extrapolate to $w \to \infty$. This yields an estimate for the mean first-passage time for an infinite width system of length L, $T_L = \lim_{w\to\infty} T(w, L)$. We then use these estimates to infer the L dependence of T_L. Owing to the exhaustive nature of this calculation, our results are limited to systems of length $L \leq 12$. However, the data are essentially exact in this range. For larger lengths, we perform an exact average over all walks and starting points, but averaged only over a finite number of configurations of large, but fixed, width. The data for $L \leq 12$ indicate that $T_L \sim L^{\alpha}$, with $\alpha = 1.49 \pm 0.03$, while the Monte Carlo data for larger L suggest a somewhat smaller value of α, but still an exponent that is larger than 4/3. This slow change in the exponent estimate may be a manifestation of the logarithmic factor in eq. (16).

Another intiguing feature of the first-passage problem is the anomalous behavior of the probability that the random walk is not trapped by time t. For a finite size system, this survival probability must asymptotically decay as $e^{-t/\tau}$, where the system size dependence of τ might be expected to be the same as T_L, if scaling holds. For our model, however, numerical calculations, based on the double extrapolation procedure outlined above, indicate that $\tau \sim L^{\beta}$, where $\beta \gtrsim 1.94$. Thus the scaling behaviors of τ and $\lim_{n\to\infty} \langle t^n \rangle^{1/n}$ are apparently different and suggest, once again, that rare event anomalies are important in determining asymptotic properties of the survival probability.

To see how such a dichotomy can arise, consider the special configuration in which the bias alternates from row to row over a width w. Although the probability that such a configuration actually occurs is proportional to e^{-aw}, the survival probability of a random walk in this configuration decreases only as e^{-t/L^2}, compared to the more rapidly decaying survival probability, $e^{-t/L^{3/2}}$, which is expected for typical configurations. It then follows that the average survival probability at sufficiently long times is dominated by the contribution from the longest-lived walks in the rare configurations. This suggests that β actually is equal to 2.

4. Isotropic random convection

The layered model has a natural generalization to an isotropic random velocity field model which also exhibits superdiffusion. One realization on the square lattice is a random "Manhattan" grid of directionalities, in which the directionality along any avenue or street is fixed along its entire length, but whose orientation is random [8]. At a coarse-grained level, this random Manhattan model can be shown to be equivalent to a random walk in a divergenceless random velocity field with power-law decay of the velocity correlation function, $\langle \boldsymbol{u}(0)\, \boldsymbol{u}(\boldsymbol{x})\rangle_c = |\boldsymbol{x}|^{-\nu}$ [16]. The mean-square displacement for the random Manhattan system can be obtained by a generalization of the arguments of eqs. (2) and (3). We formally decompose the isotropic motion into transverse and longitudinal components, and first consider the residual longitudinal bias in a typical region swept out by the transverse superdiffusive motion. Assuming $x_{\mathrm{RMS}} \sim t^{\nu}$, then from eq. (2), the mean longitudinal velocity at time t, averaged over the t^{ν} layers that a typical random walk visits, vanishes as $t^{-\nu/2}$. From eq. (3), we then conclude the $x_{\mathrm{RMS}} \sim t^{1-\nu/2}$. By isotropy, however, one must have $\nu = 1 - \nu/2$, or $\nu = 2/3$. Generalizing this argument to arbitrary spatial dimension d, yields $\nu = 2/(d+1)$ for $d < d_c = 3$, $\nu = 1/2$ for $d > d_c$, and with logarithmic corrections appearing for $d = d_c$.

For the probability distribution of displacements, even modest simulations in two dimensions indicate that eq. (9) holds over a substantial range, with $\nu = 2/3$ and $\delta = 3$, in accord with the usual relation [17] between the shape and size exponent, $\delta = (1-\nu)^{-1}$. It can be seen that rare event anomalies play a negligible role in determining the asymptotic properties of the probability distribution in this isotropic model.

5. Summary and discussion

A general mechanism for superdiffusion is the interplay between diffusion and convection by spatially inhomogeneous, but correlated, velocity fields. For the layered system, with convection along x whose magnitude is a random zero-mean function of y, the RMS longitudinal displacement increases as $t^{3/4}$, while the mean first-passage time to reach a distance x from the starting point increases as $(x \ln x)^{4/3}$. There are, however, anomalously large contributions to the tails of the underlying distribution functions for these two averages. While the natural generalization of many of these phenomena to higher dimensions is ostensibly straightforward, several issues are not yet resolved. In particular, the role of Lifshitz singularities in higher dimensions needs to be clarified. For the layered model in three dimensions, i.e., filaments of random convection in the x direction, whose orientation depends only on y and z, the natural generalization of eqs. (2) and (3) suggests that $x_{\mathrm{RMS}} \sim t^{1/2}$, as in conventional diffusion [5]. Thus it is tempting to conclude that $d = 3$ is the upper critical dimension for this model. However, the Lifshitz singularity which controls the tail of the probability distribution apparently persists to infinite dimension, and it is not yet clear how to reconcile these two disparate aspects.

As isotropic random convention model was also discussed in which the bias field takes the form of a random "Manhattan" grid. In a coarse-grained description, this bias is essentially a divergenceless velocity field, with a power-law decay of the velocity auto-correlation function. The probability distribution of displacements appears to follow a conventional, single-exponent scaling form [17] in which the shape exponent $\delta = (1 - \nu)^{-1}$, with $\nu = 2/(d+1)$ for $d < 3$ and $\nu = 1/2$ for $d \geq 3$. The first-passage characteristics of this isotropic model have not yet been investigated, and it will be interesting to see whether the exponent of the mean first-passage time will be simply related to the correlation length exponent ν.

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