Chapter 6

SPIN DYNAMICS

Kinetic spin systems play a crucial role in our understanding of non-equilibrium statistical physics. The prototypical example is the kinetic Ising model, in which the conventional Ising model of equilibrium statistical mechanics is endowed with physically-motivated transition rates that allows the system to "hop" between different microstates. Just as investigations of the equilibrium Ising model have elucidated the rich phenomenology underlying the transition between the disordered and ferromagnetically-ordered states, studies of kinetic Ising models have yielded deep insights about a host of phenomena related to the appearance of ferromagnetic order from an initially disordered paramagnetic state.

Part of the reason for the long-term interest in kinetic Ising models is their conceptual simplicity as well as their wide applicability. However, there is an even simpler kinetic spin system that is not so widely appreciated among physicists—the voter model—that will be the starting point for our discussion. The main reason for starting with this model is that it is exactly soluble in all spatial dimensions. This solution is quite interesting on its own and it also provide the framework for understanding the behavior of kinetic Ising models.

6.1 The Voter Model

The voter model was first introduced in the context of interacting particle systems. Because of its paradigmatic nature, the voter model has been one of the most extensively studied interacting particle systems. The voter model describes, in an appealing way, how consensus emerges in a population of spineless individuals. That is, each individual has no firmly fixed opinion and merely takes the opinion of one of its neighbors in an update event. As a result, a finite population of such voters eventually achieves consensus in a time that depends on the system size and on the spatial dimension. In this section, we discuss some of the most basic and striking results of the voter model. We employ physics-inspired techniques that originated from non-equilibrium statistical physics, to solve basic dynamical properties of the voter model on regular lattices in all dimensions.

In the voter model, individuals are situated at each of the sites of a graph—one for each site. This graph could be a regular lattice in d dimensions, or it could be any type of graph—such as the Erdös-Rényi random graph, or a graph with a broad distribution of degrees. Each voter can be in one of two states that, for this presentation, we label as "Democrat" and "Republican". Mathematically, the state of the voter at $\mathbf{x}$, $s(\mathbf{x})$, can take the values ± 1 only; $s(\mathbf{x}) = +1$ for a Democrat and $s(\mathbf{x}) = -1$ for a Republican.

The dynamics of the voter model is simplicity itself. Each voter has no confidence and looks to a neighbor to decide what to do. A single update event in the voter model consists of:

1. Pick a random voter.

2. The selected voter at $\mathbf{x}$ adopts the state of a randomly-selected neighbor at $\mathbf{y}$. That is, $s(\mathbf{x}) \to s(\mathbf{y})$.

3. Repeat steps 1 & 2 *ad infinitum* or stop when consensus is achieved.

Notice that a voter changes opinion only when its neighbor has the opposite opinion. A typical realization of the voter model on the square lattice is shown in Fig. 6.1, showing how the system tends to organize into single-opinion domains as time increases.

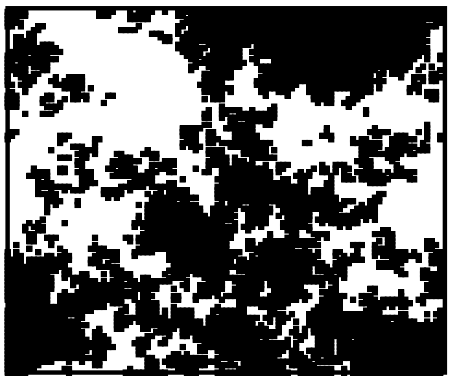

Figure 6.1: The voter model in two dimensions. Shown is a snapshot of a system on a 100×100 square lattice at time $t = 1000$, obtained by a Monte Carlo simulation. Black and white pixels denote the different opinion states.

A useful way to implement the voter model dynamics is to have each update step occur at a fixed rate. The rate at which a voter at $\mathbf{x}$ changes to the state $-s(\mathbf{x})$ may then be written as

$$w(s(\mathbf{x})) = \frac{1}{2}\left(1 - \frac{s(\mathbf{x})}{z}\sum_{\mathbf{y}\ \text{n.n.}\mathbf{x}} s(\mathbf{y})\right), \tag{6.1}$$

where the sum is over the nearest neighbors of site $\mathbf{x}$. Here z is the coordination number of the graph and we tacitly assume that each site has the same coordination number. The basic feature of this dynamical rule is that the transition rate of a voter at $\mathbf{x}$ is a linear function of the fraction of disagreeing neighbors—when a voter at $\mathbf{x}$ and all its neighbors agree, the transition rate is zero; conversely, the transition rate equals 1 if all neighbors disagree with the voter at $\mathbf{x}$. This linearity is the primary reason why the voter model is soluble. For later convenience, we define the amplitude in the transition rate so that its maximal value equals 1. It is also worth mentioning that the voter model can be generalized to include opinion changes, $s(\mathbf{x}) \to -s(\mathbf{x})$, whose rate does not depend on the local environment, by simply adding a constant to the flip rate.

To solve the voter model, we need, in principle, the probability distribution $P(\{\mathbf{s}\}, t)$ that the set of all voters are in the configuration $\{\mathbf{s}\}$ at time t. This probability distribution satisfies the master equation

$$\frac{dP(\{\mathbf{s}\})}{dt} = -\sum_{\mathbf{x}} w(s(\mathbf{x}))P(\{\mathbf{s}\}) + \sum_{\mathbf{x}} w(-s(\mathbf{x}))P(\{\mathbf{s}\}_{\mathbf{x}}). \tag{6.2}$$

Here $\{\mathbf{s}\}_{\mathbf{x}}$ denotes the state that is the same as $\{\mathbf{s}\}$ except that the voter at $\mathbf{x}$ has changed opinion. In this master equation, the loss term accounts for all possible transitions out of state $\{\mathbf{s}\}$, while the gain term accounts for transitions in which the state of one spin differs from the configuration $\{\mathbf{s}\}$. In principle, we can use this master equation to derive closed equations for all moments of the probability distribution—namely, all multiple-spin correlation functions of the form $S_{\mathbf{x},\ldots,\mathbf{y}} \equiv \langle s(\mathbf{x}) \cdots s(\mathbf{y})\rangle$ where the angle brackets denote the average $\langle f(\{\mathbf{s}\})\rangle \equiv \sum_{\mathbf{s}} f(\{\mathbf{s}\})P(\mathbf{s})$.

Let's begin by considering the simplest such correlation function, namely, the mean spin, or equivalently, the one-point function, $S(\mathbf{x}) \equiv \langle s(\mathbf{x})\rangle$. While it is possible to obtain the evolution of the mean spin and indeed any spin correlation function directly from the master equation (6.2), this approach involves some bookkeeping that is prone to error. We therefore present an alternative method that is both more direct and more instructive. In a small time interval Δt, the state of a given voter changes as follows:

$$s(\mathbf{x}, t + \Delta t) = \begin{cases} s(\mathbf{x}, t) & \text{with probability } 1 - w(s(\mathbf{x}))\Delta t, \\ -s(\mathbf{x}, t) & \text{with probability } w(s(\mathbf{x}))\Delta t. \end{cases} \tag{6.3}$$

Since the opinion at $\mathbf{x}$ changes by $-2s(\mathbf{x})$ with rate $w(s(\mathbf{x}))$, the average opinion evolves according to the rate equation

$$\frac{dS(\mathbf{x})}{dt} = -2\langle s(\mathbf{x})w(s(\mathbf{x}))\rangle.$$

Substituting in the transition rate from (6.1) and using the fact that $s(\mathbf{x})^2 = 1$, we find that for voters that are located on the sites of a d-dimensional hypercubic lattice, the rate equation has the form

$$\frac{dS(\mathbf{x})}{dt} = -S(\mathbf{x}) + \frac{1}{z}\sum_i S(\mathbf{x}+\mathbf{e}_i), \tag{6.4}$$

where $\mathbf{e_i}$ are the unit vectors of the lattice. The above evolution equation can be rewritten more compactly using the definition of the discrete Laplacian operator

$$\Delta F(\mathbf{x}) \equiv -F(\mathbf{x}) + \frac{1}{z}\sum_i F(\mathbf{x}+\mathbf{e}_i), \tag{6.5}$$

from which the mean spin evolves as $\frac{dS(\mathbf{x})}{dt} = \Delta S(\mathbf{x})$.

This rate equation shows that the mean spin performs a random walk on the lattice in continuous time. As a result, the mean magnetization, $m \equiv \sum_{\mathbf{x}} S(\mathbf{x})/N$ is conserved, as follows by summing Eq. (6.4) over all sites. There is a subtle aspect to this basic conservation law. While the magnetization of a specific system *does* change in a single update event by construction, the average over all sites and over all trajectories of the dynamics *is* conserved. The consequence of this conservation law is profound. Consider a finite system with an initial fraction ρ of Democrats and $1-\rho$ of Republicans; equivalently, the initial magnetization $m_0 = 2\rho - 1$. Ultimately, this system will reach consensus by voter model dynamics—Democrat consensus occurs with probability E and Republican consensus occurs with probability $1-E$. The magnetization of this final state is $m_\infty = E\times 1 + (1-E)\times(-1) = 2E-1$. From magnetization conservation, we obtain our first basic conclusion about the voter model: because $m_\infty = m_0$, the "exit probability" is simply $E = \rho$.

Discrete Diffusion Equation and Bessel Functions

When a random walk hops between sites of a regular but continuously in time, the master equation for the probability that the particle is on site n at time t has the generic form:

$$\dot{P}_n = \frac{\gamma}{2}(P_{n-1} + P_{n+1}) - P_n. \tag{6.6}$$

The usual case is $\gamma = 1$, corresponding to conservation of the total probability. Here we consider general values of γ because this case arises in the equations of motion for correlation functions in the kinetic Ising model. For simplicity, let us suppose that the random walk is initially at site $n = 0$. To solve this equation, we introduce the Fourier transform $P(k,t) = \sum_n P_n(t)e^{ikn}$ and find that the Fourier transform satisfies $\frac{dP(k)}{dt} = [\frac{1}{2}\gamma(e^{ik} + e^{-ik}) - 1]P(k)$. For the initial condition $P(k, t=0) = 1$, the solution is simply $P(k) = \exp[\gamma t\cos k - t]$. Now we use the generating function representation of the Bessel function, $\exp(z\cos k) = \sum_{n=-\infty}^{\infty} e^{ikn} I_n(z)$. Expanding the generating function in a power series in γt, we obtain the final result

$$P_n(t) = I_n(\gamma t)e^{-t}. \tag{6.7}$$

In the long-time limit, we use the asymptotics of the Bessel function

$$I_n(t) \sim (2\pi t)^{-1/2}\, e^t,$$

to give the asymptotic behavior

$$P_n(t) \sim \frac{1}{\sqrt{2\pi\gamma t}}\, e^{-(1-\gamma)t}.$$

Let us now solve the rate equation (6.4) explicitly for the mean spin at $\mathbf{x}$. In one dimension, this rate equation is

$$\frac{dS(x)}{dt} = -S(x) + \frac{1}{2}\left[S(x-1) + S(x+1)\right]. \tag{6.8}$$

For simplicity consider the initial condition $S(x,t=0)=\delta_{x,0}$; that is, start with a single Democrat in a background population of undecided voters. Then using the results from the above highlight on the Bessel function solution to this type of master equation, we obtain

$$S(x,t) = I_x(t)\, e^{-t} \sim \frac{1}{\sqrt{2\pi t}} \qquad \text{as } t\to\infty. \tag{6.9}$$

Exactly the same approach works in higher dimensions. Now the rate equation is

$$\frac{dS(\mathbf{x})}{dt} = -S(\mathbf{x}) + \frac{1}{z}\sum_i S(\mathbf{x}+\mathbf{e_i}), \tag{6.10}$$

where the sum is over the z nearest neighbors of $\mathbf{x}$. To solve this equation, we introduce the multidimensional Fourier transform $P(k_1,k_2,\ldots,t)=\sum_{x_1,x_2,\ldots} P_{x_1,x_2,\ldots}(t)e^{ik_1x_1}e^{ik_2x_2}\ldots$ and find that the Fourier transform in each coordinate direction factorizes. For the initial condition of one Democrat at the origin in a sea of undecided voters, the mean spin is then given by

$$S(\mathbf{x},t) = \prod_{i=1}^{d} I_{x_i}(t)\, e^{-dt} \sim \frac{1}{(2\pi t)^{d/2}}. \tag{6.11}$$

Thus the fate of a single voter is to quickly relax to the average undecided opinion of the rest of the population.

To understand how consensus is actually achieved in the voter model, we need a quantity that tells us the extent to which two distant voters agree. Such a measure is provided by the two-point correlation function, $S(\mathbf{x},\mathbf{y}) \equiv \langle s(\mathbf{x})s(\mathbf{y})\rangle$, which quantifies the extent to which two distant voters agree. Proceeding in close analogy with Eq. (6.3) the two-point function evolves as

$$s(\mathbf{x},t+\Delta t)s(\mathbf{y},t+\Delta t) = \begin{cases} s(\mathbf{x},t)s(\mathbf{y},t) & \text{with probability } 1-[w(s(\mathbf{x}))+[w(s(\mathbf{y}))]\Delta t, \\ -s(\mathbf{x},t)s(\mathbf{y},t) & \text{with probability } [w(s(\mathbf{x}))+w(s(\mathbf{y}))]\Delta t. \end{cases} \tag{6.12}$$

Thus $S(\mathbf{x},\mathbf{y})$ evolves according to

$$\frac{dS(\mathbf{x},\mathbf{y})}{dt} = -2\left\langle s(\mathbf{x})s(\mathbf{y})\big[w(s(\mathbf{x}))+w(s(\mathbf{y}))\big]\right\rangle.$$

On a hypercubic lattice, the explicit form of this rate equation is

$$\frac{dS(\mathbf{x},\mathbf{y})}{dt} = -2S(\mathbf{x},\mathbf{y}) + \sum_i \frac{1}{z}\big[S(\mathbf{x}+\mathbf{e}_i,\mathbf{y}) + S(\mathbf{x},\mathbf{y}+\mathbf{e}_i)\big]. \tag{6.13}$$

In what follows, we discuss spatially homogeneous and isotropic systems in which the correlation function depends only on the distance $r=|\mathbf{x}-\mathbf{y}|$ between two voters at $\mathbf{x}$ and $\mathbf{y}$, $G(r)\equiv S(\mathbf{x},\mathbf{y})$. Then the last two terms on the right-hand side of (6.13) are identical and this equation reduces to (6.4) apart from an overall factor of 2. At this stage, it is most convenient to consider the continuum limit, in which case Eq. (6.13) reduces to the diffusion equation

$$\frac{\partial G}{\partial t} = D\nabla^2 G, \tag{6.14}$$

with $D=2(\Delta x)^2/\Delta t$, and Δx is the lattice spacing. For the undecided initial state in which each voter is independently a Democrat or a Republican with equal probability, the initial condition is $G(r,t=0)=0$ for $r>0$. On the other hand, each voter is perfectly correlated itself, $G(0,t)=1$. We can write these two conditions succinctly as $G(r,t=0)=\delta_{r,0}$.

To understand physically how the correlation function evolves, it is expedient to work with $c\equiv 1-G$; c also satisfies the diffusion equation, but now with the initial condition $c(r>a,t=0)=1$. The appropriate boundary condition for the continuum limit is $c(r=a,t)=1$; that is, the absorbing point at the origin is replaced by a small absorbing sphere of non-zero radius a. One should think of a as playing the role of the lattice spacing; a non-zero radius is needed so that a diffusing particle can actually hit the sphere. Physically,

then, we study how an initially constant density profile evolves in the presence of a small absorbing sphere at the origin.

The exact solution for this concentration profile can be easily obtained in the Laplace domain. Laplace transforming the diffusion equation gives $sc - 1 = D\nabla^2 c$; the inhomogeneous term arises from the constant-density initial condition. A particular solution to the inhomogeneous equation is simply $c = 1/s$, and the homogeneous equation

$$c'' + \frac{d-1}{r}\, c' - \frac{s}{D}\, d = 0$$

has the general solution $c = Ar^\nu I_\nu(r\sqrt{s/D}) + Br^\nu K_\nu(r\sqrt{s/D})$, where I_ν and K_ν are the modified Bessel functions of order ν, with $\nu = (2-d)/2$. Since the concentration is finite as $r \to \infty$, the term with I_ν must be rejected. Then matching to the boundary condition $c = 0$ at $r = a$ gives

$$c(r,s) = \frac{1}{s}\left[1 - \left(\frac{r}{a}\right)^\nu \frac{K_\nu(r\sqrt{s/D})}{K_\nu(a\sqrt{s/D})}\right]. \tag{6.15}$$

For spatial dimension $d > 2$ ($\nu < 0$), we use $K_\nu = K_{-\nu}$ and the small-argument form $K_\nu(x) \propto (2/x)^\nu$ to give the leading small-s behavior

$$c(r, s \to 0) = \frac{1}{s}\left[1 - \left(\frac{a}{r}\right)^{d-2}\right].$$

Thus in the time domain, the concentration profile approaches the static electrostatic solution, $c(r) = 1 - (a/r)^{d-2}$! A steady state is achieved because there is a non-zero probability that a diffusing particle never hits the absorbing sphere (transience). The depletion of the concentration near the sphere is sufficiently slow that it is replenished by re-supply from more distant particles. In terms of the voter model, the two-particle correlation function asymptotically becomes $G(r) \to (a/r)^{d-2}$ for $d > 2$. Thus the influence of one voter on a distant neighbor decays as a power law in their separation.

Now let's study the case $d \leq 2$ ($\nu \geq 0$). Here a diffusing particle eventually hits the sphere (recurrence; see Sec. 2.3), leading to a growing depletion zone about the sphere. While the time dependence of c can be obtained by inverting the Laplace transform in Eq. (6.15), we present a simpler and physically-driven approach—the *quasi-static approximation*. The basis of this simple and generally quite useful approximation is the observation that even though the concentration profile evolves in the depletion zone, the change is sufficiently slow that we can again obtain the solution from Laplace's equation. This Laplacian solution must then match with the unperturbed concentration for $r > \sqrt{Dt}$. This matching is the mechanism by which a time dependence arises. Thus we solve the Laplace equation in the intermediate range of $a < r < \sqrt{Dt}$ and match to the static value $c = 1$ when $r\sqrt{Dt}$. From the generic solution $c(r) = A + B/(r)^{d-2}$, together with the boundary conditions $c(a) = 0$ and $c(\sqrt{Dt}) = 1$, we obtain

$$c(r,t) = \frac{1 - (a/r)^{d-2}}{1 - (a/\sqrt{Dt})^{d-2}} \to \left(\frac{\sqrt{Dt}}{r}\right)^{d-2} \qquad t \to \infty.$$

For $d = 2$, the same quasi-static approach still works. We use the generic solution to the Laplace equation for $d = 2$, namely, $c(r) = A + B\ln r$ and apply the boundary conditions at $r = a$ and $r = \sqrt{Dt}$ to obtain

$$c(r,t) = \frac{\ln(r/a)}{\ln(\sqrt{Dt}/a)} \to \frac{\ln r}{\ln t} \quad t \to \infty.$$

In summary, the two-spin correlation function for $r > a$ has the asymptotic behavior for general spatial dimensions:

$$G(r,t) \sim \begin{cases} 1 - \left(\dfrac{r}{\sqrt{Dt}}\right)^{2-d} & d < 2 \text{ and } 0 < r < \sqrt{Dt}; \\[2ex] 1 - \dfrac{\ln(r/a)}{\ln(\sqrt{Dt}/a)} & d = 2 \text{ and } a < r < \sqrt{Dt}; \\[2ex] \left(\dfrac{a}{r}\right)^{d-2} & d > 2 \text{ and } a < r. \end{cases} \tag{6.16}$$

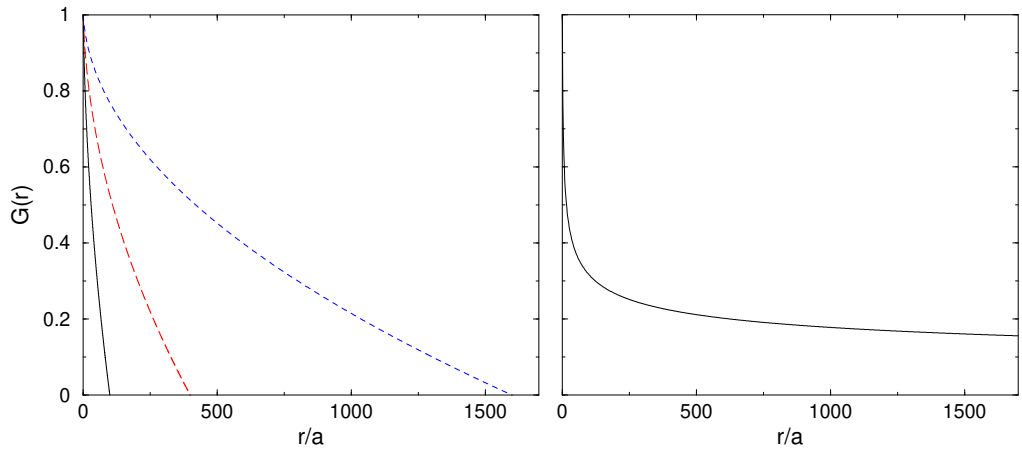

Figure 6.2: The spin correlation function for the voter model from Eq. vm-corr-sum for $d < 2$ (left), for progressively later times, and the steady-state correlation function for $d > 2$ (right).

An important feature for $d \leq 2$ is that the correlation function at fixed r approaches 1—distant spins gradually become more strongly correlated. This feature is a manifestation of *coarsening* in which the voters organize into a mosaic of single-opinion enclaves whose characteristic size increases with time. As we shall see, coarsening typifies many types of phase-ordering kinetics. On the other hand, for $d > 2$ the voter model approaches a steady state and there is no coarsening in the spatial arrangement of the voters.

There are two important consequences for the voter model that can be deduced from the behavior of the correlation function. The first is that we can immediately determine the time dependence of the density of "interfaces", namely, the fraction n of neighboring voters of the opposite opinion. As shall use extensively later on, it is helpful to represent an interface as an effective particle that occupies the bond between two neighboring voters of the opposite opinion. This quantity provides the right way to characterize the departure of system from consensus. For two nearest-neighbor sites $\mathbf{x}$ and $\mathbf{y}$, we relate the correlation function to the interface density by

$$\begin{aligned} G(\mathbf{x},\mathbf{y}) = \langle s(\mathbf{x})s(\mathbf{y})\rangle &= [\text{prob}(++) + \text{prob}(--)] - [\text{prob}(+-) + \text{prob}(-+)] \\ &= 1 - n \quad - \quad n = 1 - 2n. \end{aligned} \tag{6.17}$$

Thus the density of interfaces is related to the near-neighbor correlation function via $n = (1 - G(\mathbf{x},\mathbf{y}))/2$. Using our result (6.16) for the correlation function, the time dependence of the interfacial density is then

$$n(t) \sim \begin{cases} t^{d/2-1} & d < 2, \\ 1/\ln t & d = 2, \\ \mathcal{O}(1) & d > 2. \end{cases} \tag{6.18}$$

When $d \leq 2$, the probability of having two voters with opposite opinions asymptotically vanishes and the system develops a coarsening mosaic of single-opinion domains (Fig. 6.1). At the marginal dimension of $d = 2$ the coarsening process is very slow and the density of interfaces asymptotically vanishes as $1/\ln t$. In higher dimensions, the system reaches a dynamic frustrated state where voters of opposite opinion coexist and continually evolve such that the mean density of each type of voter remains fixed.

The second basic consequence that follows from the correlation function is the time T_N to reach consensus for a finite system of N voters. For this estimate of the consensus time, we use the fact that the influence of any voter spread diffusively through the system. Thus starting with some initial state, the influence range of one voter is of the order of $\sqrt{Dt}$. We then define consensus to occur when the total amount of correlation within a distance of $\sqrt{Dt}$ of a particular voter equals the total number of voters N. The consensus criterion

therefore becomes

$$\int^{\sqrt{Dt}} G(r)\, r^{d-1}\, dr = N. \tag{6.19}$$

The lower limit can be set to 0 for $d = 1$ and should be set to a for $d > 1$. Substituting the expressions for the correlation function given in Eq. (6.16) into this integral, the time dependence can be extracted merely by scaling and we find the asymptotic behavior

$$T_N \propto \begin{cases} N^{2/d} & d < 2; \\ N \ln N & d = 2; \\ N & d > 2. \end{cases}$$

Thus as the dimension decreases below 2, consensus takes a progressively longer to achieve. This feature reflects the increasing difficulty in transmitting information when the dimensionality decreases.

Let us now derive the exact solution for the correlation function without using the continuum approximation. This solution is nothing more than the lattice Green's function for the diffusion equation. It is convenient to rescale the time variable by 2, $\tau = 2t$, so that the correlation function satisfies precisely the same equation of motion as the average magnetization

$$\frac{d}{d\tau} G(\mathbf{x}) = -G(\mathbf{x}) + \frac{1}{z} \sum_i G(\mathbf{x} + \mathbf{e}_i). \tag{6.20}$$

We consider the uncorrelated initial condition $G(\mathbf{x}, 0) = \delta(\mathbf{x})$ and the boundary condition is $G(\mathbf{0}) = 1$. The evolution equation and the initial conditions are as for the autocorrelation function where the solution is $I_{\mathbf{m}}(\tau) e^{-d\,\tau}$. Since the equation is linear, every linear combination of these "building-blocks" is also a solution. Therefore, we consider the linear combination

$$G(\mathbf{x}, \tau) = I_{\mathbf{x}}(\tau) e^{-d\,\tau} + \int_0^\tau d\tau' J(\tau - \tau') I_{\mathbf{x}}(\tau') e^{-d\,\tau'}. \tag{6.21}$$

The kernel of the integral is identifies as a source with strength $\delta(\tau) + J(\tau)$. This source is fixed by the boundary condition:

$$1 = \left[I_0(\tau) e^{-\tau}\right]^d + \int_0^\tau d\tau'\, J(\tau - \tau') [I_0(\tau') e^{-\tau'}]^d. \tag{6.22}$$

We are interested in the asymptotic behavior of the correlation function. This requires the $\tau \to \infty$ behavior of the source term. Thus, we introduce the Laplace transform $\hat{J}(s) = \int_0^\infty d\tau\, e^{-s\tau} J(\tau)$. Exploiting the convolution structure of the integral yields

$$\hat{J}(s) = [s\hat{I}(s)]^{-1} - 1 \qquad \text{with} \qquad \hat{I}(s) = \int_0^\infty d\tau\, e^{-s\tau} [I_0(\tau) e^{-\tau}]^d. \tag{6.23}$$

Using the integral representation of the Bessel function, $I_0(\tau) = \int_0^{2\pi} \frac{dq}{2\pi} e^{\tau \cos q}$, the latter transform is expressed as an integral

$$\hat{I}(s) = \int_0^{2\pi} \frac{d\mathbf{q}}{(2\pi)^d}\, \frac{1}{s + \sum_{i=1}^d (1 - \cos q_i)}. \tag{6.24}$$

The $\tau \to \infty$ asymptotic behavior of the source and the correlation function is ultimately related to the $s \to 0$ asymptotic behavior of this integral. The integral diverges, $\hat{I}(s) \sim s^{d/2-1}$, when $d < 2$, but it remains finite when $d > 2$. The leading $s \to 0$ behavior of the Laplace transform is therefore

$$\hat{J}(s) \sim \begin{cases} s^{-d/2} & d < 2, \\ s^{-1} \ln s^{-1} & d = 2, \\ s^{-1} & d > 2. \end{cases} \tag{6.25}$$

6.2 Glauber Model in One Dimension

The Ising model provides an appealing and universal description of phase transitions in ferromagnets. Owing to its conceptual simplicity and broad applicability, statistical mechanical studies of the Ising model played a fundamental role in the development of the modern theory of critical phenomena. In the Ising model, a regular lattice is populated by 2-state spins that may take one of two values: $s(\mathbf{x}) = \pm 1$. Pairs of nearest-neighbor spins experience a ferromagnetic interaction that favors their alignment. The Hamiltonian of the system is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j, \tag{6.26}$$

where the sum is over nearest neighbors (i, j) on the lattice. Every parallel pair of neighboring spins contributes $-J$ to the energy and every antiparallel pair contributes $+J$. When the coupling constant is positive, the interaction favors ferromagnetic order.

The main feature of the Ising model is that ferromagnetism appears spontaneously in the absence of any driving field when the temperature T is less than a critical temperature T_c and the spatial dimension $d > 1$. Above T_c, the spatial arrangement of spins is spatially disordered, with equal numbers of spins in the states $+1$ and -1. Consequently, the magnetization is zero and spatial correlations between spins decay exponentially with their separation. Below T_c, the magnetization is non-zero and distant spins are strongly correlated. All thermodynamic properties of the Ising model can be obtained from the partition function $Z = \sum \exp(-\beta \mathcal{H})$, where the sum is over all spin configurations of the system, with $\beta = 1/k_B T$ and k_B is the Boltzmann constant.

While equilibrium properties of the Ising model follow from the partition function, its non-equilibrium properties depend on the nature of the spin dynamics. There is considerable freedom in formulating this dynamics that is dictated by physical considerations. For example, the spins may change one at a time or in correlated blocks. More fundamentally, the dynamics may or may not conserve the magnetization. The role of a conservation law depends on whether the Ising model is being used to describe alloy systems, where the magnetization (related to the composition of the material) is necessarily conserved, or spin systems, where the magnetization does not have to be conserved. This lack of uniqueness of dynamical rules is generic in non-equilibrium statistical physics and it part of the reason why there do not exist universal principles, such as free energy minimization in equilibrium statistical mechanics, that prescribe how to solve a non-equilibrium spin system.

Spin evolution

In this section, we discuss a particular realization of the kinetic Ising model that evolves by non-conservative single-spin-flip dynamics. This model, first introduced by Glauber in 1963, represents a simple way to extend the Ising model to non-equilibrium situations. Here we will focus on the one-dimensional system, as this model is exactly soluble by analytical methods. Later on we will study the Ising-Glauber model in higher dimensions, and well as different types of spin dynamics, including conservative Kawasaki spin-exchange dynamics, and cluster dynamics, in which correlated blocks of spins flip simultaneously. In the Glauber model, spins are selected one at a time in random order and each changes at a rate that depends on the change in the energy of the system as a result of this update. Because only single spins can change sign in an update, $s_j \to -s_j$, where s_j is the spin value at site j, the magnetization is generally *not* conserved.

There are three types of transitions that can arise when a single spin flips: energy raising, energy lowering, and energy neutral transitions (Fig. 6.3). Energy raising events occur when a spin is aligned with a majority of its neighbors and *vice versa* for energy lower events. Energy conserving events occur when the net magnetization of the neighbors is zero. The basic principle to fix the rates of the various types of events is the *detailed balance condition*. Mathematically, this condition is:

$$P(\{s\})w(s \to s'_j) = P(\{s'_j\})\, w(s'_j \to s). \tag{6.27}$$

Here $\{s\}$ denotes the state of all the spins in the system, $\{s'_j\}$ denotes the state derived from $\{s\}$ in which the spin at i is flipped, and $w(s \to s'_j)$ denotes the transition rate from $\{s\}$ to $\{s'_j\}$.

The meaning of the detailed balance condition is simple. In the abstract space of all 2^N possible spin states of a system of N spins, Glauber dynamics connects states which differ by the reversal of a single spin.

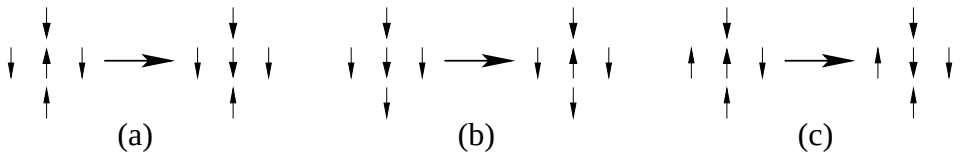

Figure 6.3: (a) Energy lowering, (b) energy raising, and (c) energy conserving spin-flip events on the square lattice.

When detailed balance holds, the probability currents from state $\{s\}$ to $\{s'_j\}$ and from $\{s'_j\}$ to $\{s\}$ (the left and right sides of Eq. (6.27)) are equal so there is no net probability current across any link in this state space. If $P(\{s\})$ are the equilibrium Boltzmann weights, then the transition rates defined by Eq. (6.27) ensure that any initial spin state will eventually relax to the equilibrium thermodynamic equilibrium state for any non-zero temperature. Thus dynamics that satisfy detailed balance are required if one seeks to understand how equilibrium is approached when a system is prepared in an out of equilibrium state.

In the following discussion of analytical methods, we specialize to the case of one dimension. Here the detailed balance condition is sufficient to fix the actual flip rates. Following Glauber, we assume that the flip rate of the $j^{\rm th}$ spin depends on the neighbors with which there is a direct interaction, namely, s_j and $s_{j\pm1}$. For an isotropic system, the rate should have left/right symmetry (invariance under the interchange $i+1 \leftrightarrow i-1$) and up/down symmetry (invariance under the reversal of all spins).[1]. For a homogeneous one-dimensional system, these conditions constrain the rate to have the form $w(s \to s'_j) = A + Bs_j(s_{j-1}+s_{j+1})$. This flip rate is simply the energy of the $i^{\rm th}$ spin up to an additive constant. We now write this flip rate in the following suggestive form

$$w(s \to s'_j) = \frac{\alpha}{2}\left[1 - \frac{\gamma}{2}s_j(s_{j-1}+s_{j+1})\right] = \begin{cases} \frac{\alpha}{2}(1-\gamma) & \text{for spin state } \uparrow\uparrow\uparrow \text{ or } \downarrow\downarrow\downarrow; \\ \frac{\alpha}{2} & \text{for spin state } \uparrow\uparrow\downarrow \text{ or } \downarrow\downarrow\uparrow; \\ \frac{\alpha}{2}(1+\gamma) & \text{for spin state } \uparrow\downarrow\uparrow \text{ or } \downarrow\uparrow\downarrow\,. \end{cases} \tag{6.28}$$

When the two neighbors are antiparallel (no local field), the flip rate is simply a constant that we take to be 1/2 ($\alpha = 1$) without loss of generality. For $\gamma > 0$, the flip rate favors aligning s_j with its neighbors and *vice versa* for $\gamma < 0$.

We now fix γ by exploiting detailed balance:

$$\frac{w(s \to s'_j)}{w(s'_j \to s)} = \frac{1 - \frac{\gamma}{2}s_j(s_{j-1}+s_{j+1})}{1 + \frac{\gamma}{2}s_j(s_{j-1}+s_{j+1})} = \frac{P(\{s'_j\})}{P(\{s\})} = \frac{e^{-\beta J\epsilon_j}}{e^{+\beta J\epsilon_j}}, \tag{6.29}$$

with $\epsilon_j \equiv -s_j(s_{j-1}+s_{j+1})$. We simplify the last quantity by exploiting the ±1 algebra of Ising spins to write

$$\frac{e^{-\beta J\epsilon_j}}{e^{+\beta J\epsilon_j}} = \frac{\cosh(-\beta J\epsilon_j) + \sinh(-\beta J\epsilon_j)}{\cosh\beta J\epsilon_j + \sinh\beta J\epsilon_j} = \frac{1 - \tanh(2\beta J\frac{\epsilon_j}{2})}{1 + \tanh(2\beta J\frac{\epsilon_j}{2})} = \frac{1 - \frac{1}{2}\epsilon_j \tanh 2\beta J}{1 + \frac{1}{2}\epsilon_j \tanh 2\beta J},$$

where in the last step we use the fact that $\tanh ax = a\tanh x$ for $a = 0, \pm1$. Comparing with Eq. (6.28), we deduce that $\gamma = \tanh 2\beta J$. Thus the flip rate is

$$w(s_j) = \frac{1}{2}\left[1 - \frac{1}{2}\tanh 2\beta J\, s_j(s_{j-1}+s_{j+1})\right]. \tag{6.30}$$

For $T \to \infty$, $\gamma \to 0$ and all three types of spin-flip events shown in Eq. (6.28) are equiprobable. Conversely, for $T \to 0$, $\gamma \to 1$, and energy raising spin-flip events are prohibited.

The probability distribution $P(\{\mathbf{s}\}, t)$ that the system has the microscopic spin configuration $\mathbf{s}$ at time t satisfies the same master equation (6.2) as the voter model. Consequently, the equation of motion for the

[1] Actually the most general rate that satisfies the constraints of locality within the interaction range, symmetry, and isotropy is $w_(s_j) = (1/2)(1 + \delta s_{j-1}s_{j+1})\left[1 - (\gamma/2)s_j(s_{j-1}+s_{j+1})\right]$

low-order correlation functions are:

$$\frac{dS_j}{dt} = -2\big\langle s_j w(s_j)\big\rangle, \tag{6.31a}$$

$$\frac{dS_{i,j}}{dt} = -2\big\langle s_i s_j [w(s_i) + w(s_j)]\big\rangle, \tag{6.31b}$$

where the subscripts i and j denote the $i^{\rm th}$ and $j^{\rm th}$ site of a one-dimensional lattice.

Using the transition rates given in (6.30) and the identity $s_j^2 = 1$, the rate equation for the average spin S_j is

$$\frac{dS_j}{dt} = -S_k + \frac{\gamma}{2}\left(S_{j-1} + S_{j+1}\right). \tag{6.32}$$

With the initial condition $S_j(0) = \delta_{j,0}$, the solution is (see the highlight on the Bessel function solution to discrete diffusion)

$$S_j(t) = I_j(\gamma t) e^{-t}. \tag{6.33}$$

The new feature compared to the corresponding voter model solution is the presence of the temperature-dependent factor γ. Now the average spin at any site decays as $S_j(t) \sim (2\pi\gamma t)^{-1/2} e^{-(1-\gamma)t}$. For $T > 0$, the decay is exponential in time, $S_j \sim e^{-t/\tau}$, with relaxation time $\tau = (1-\gamma)^{-1}$, while for $T = 0$ the decay is algebraic in time, $S_j \simeq (2\pi t)^{-1/2}$. The magnetization $m = N^{-1}\sum_j S_j$ satisfies $\frac{dm}{dt} = -(1-\gamma)m$, so that m decays exponentially with time at any positive temperature,

$$m(t) = m(0) e^{-(1-\gamma)t}, \tag{6.34}$$

and is conserved at zero temperature, just as in the voter model. The Ising-Glauber in one dimension model nicely illustrates critical slowing down—slower relaxation at the critical point ($T = 0$ in one dimension) than for $T > 0$.

The mean spin can also be directly solved for a general initial condition, $S_j(t=0) = \sigma_j$, with σ_j an arbitrary function between $+1$ and -1. Then the Fourier transform of the initial condition is $s_k(t=0) = \sum_n \sigma_n e^{ikn}$. Using this result, the Fourier transform of the solution to the equation of motion (6.32) is

$$S_k(t) = S_k(t=0) e^{(\gamma\cos k - 1)t} = \sum_m e^{ikm}\sigma_m \sum_n I_n(\gamma t) e^{ikn}\, e^{-t}.$$

Now define $\ell = m + n$ to recast the exponential factors as a single sum to facilitate taking the inverse Fourier transform:

$$S_k(t) = \sum_\ell e^{ik\ell} \sum_m \sigma_m\, I_{\ell-m}(\gamma t)\, e^{-t}.$$

From the expression above we may simply read off the solution as the coefficient of $e^{ik\ell}$:

$$S_\ell = \sum_m \sigma_m\, I_{\ell-m}(\gamma t)\, e^{-t}. \tag{6.35}$$

As we shall encounter next, this solution is useful for solving the two-spin correlation function.

Let's now study the pair correlation function, $S_{i,j} = \langle s_i s_j\rangle$. As a preliminary, it is useful to highlight a geometrical equivalence between the kinetic Ising model and diffusion-limited reactions. As given by Eq. (6.17), there is a one-to-one mapping between a spin configuration and an arrangement of domain wall quasi particles. Two neighboring antiparallel spins are equivalent to a domain wall that is halfway between the two spins, while two neighboring parallel spins has no intervening domain wall (Fig. 6.4). Energy raising spin flips are equivalent to creating a nearest-neighbor pair of domain walls, while energy lowering moves correspond to annihilation of two neighboring walls. Energy conserving flips correspond to the hopping of a domain wall between neighboring sites. At $T = 0$, where domain wall creation is forbidden, Ising-Glauber

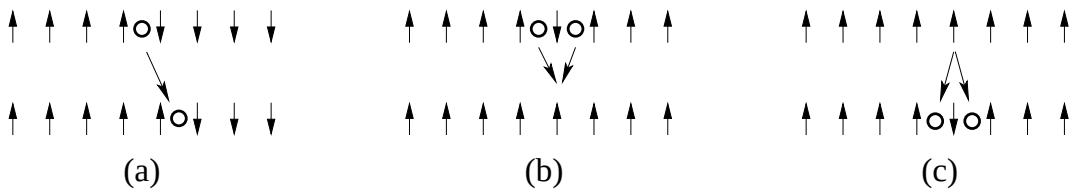

Figure 6.4: Mapping between states of the Ising or the voter models in one dimension and domain wall particles between neighboring pairs of antiparallel spins. Shown are the equivalences between: (a) an energy conserving move and diffusion of a domain wall, (b) energy lowering moves and annihilation of two domain walls, and (c) energy raising moves and creation of a pair of domain walls.

kinetics is then equivalent to *irreversible diffusion-controlled annihilation*, $A + A \to 0$. In this process a population of identical particles diffuses freely and mutual annihilation occurs when two particle come into contact. Similarly, for $T > 0$, Ising-Glauber kinetics is equivalent to diffusion-controlled annihilation with pair creation. As we will see, we can use the known results about the Ising-Glauber kinetics to infer the time dependence of the particle density in the corresponding reaction processes.

We focus on translationally invariant systems where the correlation function depends only the separation of the two spins, $G_k \equiv S_{i,i+k}$. The master equation (6.31b) becomes

$$\frac{dG_k}{dt} = -2G_k(t) + \gamma\,(G_{k-1} + G_{k+1}) \tag{6.36}$$

for $k > 0$. This equation needs to be supplemented by the boundary condition $G_0(t) = 1$. Thus the pair correlation function evolves in nearly the same way as the mean spin. However, because of the existence of the fixed boundary condition at the origin, the master equation also admits an exponential equilibrium solution. that is determined by assuming that $G_k(\infty) \propto \eta^k$ and substituting this form into Eq. (6.36) with the left-hand side set to zero. These steps lead to the following condition for η: $2\gamma = \eta + \eta^{-1}$ whose solution is $\eta = [1 - \sqrt{1-\gamma^2}]/\gamma = \tanh \beta J$. The equilibrium pair correlation function therefore decays exponentially in the distance between the two spins,

$$G_k(\infty) = e^{-k/\xi}\,, \tag{6.37}$$

with correlation length $\xi^{-1} = 1/\ln(\coth \beta J)$. This result coincides with the correlation function obtained directly from thermodynamics. As expected, the correlation length ξ diverges as $T \to 0$, indicative of a phase transition, and ξ vanishes at infinite temperature.

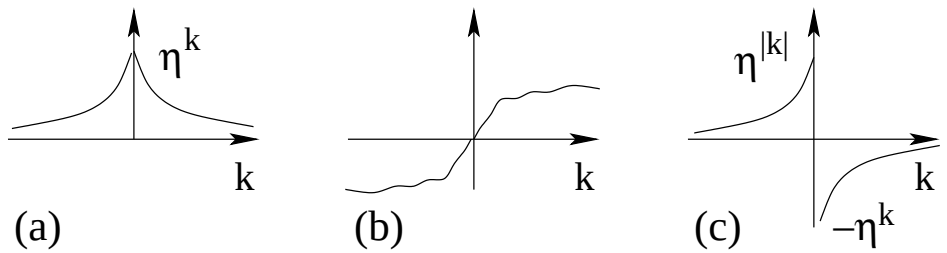

Figure 6.5: (a) Equilibrium correlation function and (b) an arbitrary antisymmetric initial condition. To find $G_k(t)$ for $k > 0$, we superpose the solutions for the three initial conditions shown. This superposition satisfies both the initial condition and the boundary condition.

We now discuss how to solve the time dependence of the correlation function with a prescribed initial condition $G_k(t = 0)$ and the boundary condition $G_0 = 1$. Since the master equation for the correlation function has the same form as that for the mean spin apart from an overall factor of 2, the general solution will be built from components of the form as (6.35) with the replacement of $\gamma \to 2\gamma$. We now need to determine the appropriate linear combination of these component solutions that simultaneously satisfy the

initial condition $G_k(t=0)$ and the boundary conditions. One piece of the full solution is just the equilibrium correlation function $G_k(\infty) = \eta^{|k|}$. To this we add the general homogeneous solution that satisfies the prescribed constraints. Pictorially, the appropriate initial condition for the homogeneous solution consists of an arbitrary odd function plus an antisymmetric piece that cancels the equilibrium solution for $k > 0$ (Fig. 6.5. The antisymmetry of these pieces ensure that $G_0 = 1$ and that the prescribed initial condition is satisfied for $k > 0$.

The general solution for $k > 0$ therefore is:

$$\begin{aligned} G_k(t) &= \eta^k + e^{-2t} \sum_{\ell=-\infty}^{\infty} G_\ell(0) I_{k-\ell}(2\gamma t) \\ &= \eta^k + e^{-2t} \sum_{\ell=1}^{\infty} [G_\ell(0) - \eta^\ell] I_{k-\ell}(2\gamma t) + e^{-2t} \sum_{\ell=-1}^{-\infty} [G_\ell(0) + \eta^{|\ell|}] I_{k-\ell}(2\gamma t) \\ &= \eta^k + e^{-2t} \sum_{\ell=1}^{\infty} [G_\ell(0) - \eta^\ell][I_{k-\ell}(2\gamma t) - I_{k+\ell}(2\gamma t)]. \end{aligned} \tag{6.38}$$

We restrict ourselves to the case of $T = 0$, where two special cases lead to nice results:

1. Antiferromagnetic initial state, $G_k(0) = (-1)^k$. In this case, every site of the dual lattice is initially occupied by domain wall particle. For this initial state, the nearest-neighbor correlation function in Eq. (6.38) reduces to

$$G_1(t) = 1 - 2e^{-2t} \sum_{j\,\text{odd}} [I_{1-j}(2t) - I_{1+j}(2t)] = 1 - 2e^{-2t}\, I_0(2t),$$

   where we have used $I_n = I_{-n}$.

2. Random initial state, $G_k(0) = m_0^2$, where m_0 is the initial magnetization. Then the nearest-neighbor correlation function is

$$G_1(t) = 1 = e^{-2t}(m_0^2 - 1) \sum_j [I_{1-j}(2t) - I_{1+j}(2t)] = 1 - 2e^{-2t}\,(m_0^2 - 1)\,[I_0(2t) + I_1(2t)]\,.$$

From these two solutions, the domain wall densities are

$$\rho(t) = \frac{1 - G_1}{2} = \begin{cases} I_0(2t)\, e^{-2t} \sim \dfrac{1}{\sqrt{4\pi t}} & \text{antiferromagnetic,} \\ \dfrac{1 - m_0^2}{2}\,[I_0(2t) + I_1(2t)]\; e^{-2t} \sim \dfrac{1 - m_0^2}{\sqrt{4\pi t}} & \text{uncorrelated.} \end{cases} \tag{6.39}$$

If the initial magnetization $m_0 = 0$ for the random initial condition, then the asymptotic domain wall density universally vanishes as

$$\rho(t) \sim (4\pi t)^{-1/2}, \tag{6.40}$$

independent of the initial domain wall density! Because the number of domain walls decrease with time, their separation correspondingly increases. The system therefore coarsens, as domains of parallel spins grow with the diffusive length scale $t^{1/2}$. A final important point is that Eq. (6.39) also represents the exact solution for diffusion-limited annihilation $A + A \to 0$!

Domain length distribution

In the previous section, we obtained the density of domain walls or alternatively, the average domain size. Now we ask the more fundamental question: what is the distribution of domain sizes in a one-dimensional system of length L? Let P_k be the probability to find a domain of size k, namely, a configuration in which of k consecutive spins are aligned and the two spins at the ends of this string are both oppositely oriented

to the string. To have a system-size independent quantity, we define this probability per unit length. We now seek the time dependence of this domain size distribution.

We can obtain partial information about this distribution from basic physical considerations. For example, the domain wall density ρ, which scales as $t^{-1/2}$, is given by $\sum_k P_k$, while the domain size distribution obeys the normalization condition $\sum_k kP_k = 1$. Further, from the the diffusive nature of the evolution, the only physical length scale grows as $t^{1/2}$. These facts suggest that the domain size distribution has the scaling form

$$P_k(t) \simeq t^{-1}\Phi(kt^{-1/2}). \tag{6.41}$$

Here the prefactor ensures that the mean domain size (per unit length) is fixed to one, that is, $\int x\Phi(x)dx = 1$, while the asymptotic decay of the total density (6.40) gives the condition $\int \Phi(x)dx = (4\pi)^{-1/2} \equiv C$.

By simple physical reasoning, we can also infer the short-distance tail of the scaling function $\Phi(x)$ from the long-time decay of the domain density. Consider the role of the shortest possible domain (of length 1) on rate equation for the domain density ρ. When a domain that consists of single spin flips, three domains merge into a single larger domain illustrated below:

$$\cdots\downarrow\underbrace{\uparrow\cdots\uparrow\uparrow}\underline{\downarrow}\underbrace{\uparrow\uparrow\cdots\uparrow}\downarrow\cdots \xrightarrow{1} \cdots\downarrow\underbrace{\uparrow\cdots\uparrow\uparrow\underline{\uparrow}\uparrow\uparrow\cdots\uparrow}\downarrow\cdots .$$

Since such events, in which two domains disappear, occur with a unit rate, the domain density decays according to

$$\frac{d\rho}{dt} = -2P_1. \tag{6.42}$$

Using Eq. (6.40), we obtain $P_1 \sim \frac{C}{4}\,t^{-3/2}$. On the other hand, expanding Φ in a Taylor series gives $P_1 \cong \Phi(0)t^{-1} + \Phi'(0)t^{-3/2} + \cdots$. Comparing these two results, we deduce that $\Phi(0) = 0$ and $\Phi'(0) = \frac{C}{4}$. Therefore the scaling function vanishes linearly in the small-argument limit:

$$\Phi(x) \sim \frac{C}{4}\,x, \qquad \text{as} \qquad x \to 0. \tag{6.43}$$

This linear decrease in the small-size tail of the probability distribution is a generic feature of many one-dimensional interacting many-body systems.

While scaling arguments provide considerable information about the asymptotic behavior of the size distribution, we are interested in the distribution itself. The exact solution is not yet known, and we present an approximate solution that is based on the *Independent Interval Approximation* that correctly describes the main qualitative aspects of the domain size distribution. The basis of this approximation is the assumption that the sizes of neighboring domains are uncorrelated, an assumption makes the the domain size distribution analytically tractable. As we shall see, this approach can be applied to a variety of one-dimensional domain evolution and reaction processes.

Under the assumption that the sizes of neighboring domains are uncorrelated, the domain distribution obeys a closed set of master equations. In an infinitesimal time interval Δt, the size distribution changes as follows:

$$\begin{aligned} P_k(t+\Delta t) - P_k(t) = -2\Delta t P_k(t) + \Delta t\, P_{k+1} + \Delta t\, P_{k-1}\left(1 - \frac{P_1}{\rho}\right) \\ + \Delta t\, P_1 \sum_{i+j=k-1} \frac{P_i}{\rho}\frac{P_j}{\rho} - \Delta t\, P_1 \frac{P_k}{\rho}\,. \end{aligned} \tag{6.44}$$

The first line accounts for length changes due to a domain wall hopping by ± 1. These events are equivalent to an effective diffusion of a single domain. The factor $1 - P_1/\rho$ ensures that the neighboring domain has length greater than one so that there is no possibility that two domain walls meet. The second line accounts for changes in the domain distribution due to merging of domains. Because any merger requires the presence of a domain of length one, the terms that account for such events are proportional to P_1. The gain term accounts for the merger of three domains of lengths i, j, and 1, with $i + j + 1 = k$. The loss term accounts

for the merger of a domain of size k with a domain of any other size. Taking the limit $\Delta t \to 0$ then yields the master equation

$$\frac{dP_k}{dt} = -2P_k + P_{k-1} + P_{k+1} + \frac{P_1}{\rho^2} \sum_{i+j=k-1} P_i\, P_j - \frac{P_1}{\rho}(P_{k-1} + P_k). \tag{6.45}$$

These master equations equations apply for any $k \geq 1$, subject to the boundary condition $P_0 = 0$. From the master equation, one can check that the total density $\rho = \sum_k P_k$ indeed satisfies the exact equation (6.42) and that $\sum_k k\, dP_k/dt = 0$. Since the typical domain size grows indefinitely, it is sensible to treat k as a continuous variable. In this limit, we replace the integer k by the real-valued variable x so that the diffusive terms in the master equation are simply replaced by the Laplacian. Using $\rho \simeq Ct^{-1/2}$, $P_1 \simeq \frac{C}{4}t^{-3/2}$, and the scaling form (6.41) then gives the integro-differential equation for the scaling function

$$\frac{d^2\Phi}{dx^2} + \frac{1}{2}\frac{d(x\Phi)}{dx} + \frac{1}{4C}\int_0^x \Phi(y)\Phi(x-y)\, dy = 0. \tag{6.46}$$

Given the convolution in the last term of this master equation, we introduce the Laplace transform, $\phi(s) = C^{-1}\int_0^\infty \Phi(x)e^{-sx}dx$, to reduce the integro-differential equation to the ordinary nonlinear differential equation

$$\frac{d\phi}{ds} = \frac{\phi^2}{2s} + 2s\,\phi - \frac{1}{2s}, \tag{6.47}$$

with the boundary condition $\phi(0) = 1$. Since we know the first two terms in the series expansion of Φ, the small-s behavior $\phi(s)$ is given by $\phi(s) \approx 1 - C^{-1}s + \cdots$. Moreover, the linear behavior (6.43) implies the decay $\phi(s) \simeq (4s^2)^{-1}$ as $s \to \infty$.

Eq. (6.47) is a Riccati equation and it can be reduced to the second-order linear equation

$$\frac{d^2\psi}{ds^2} + \frac{d\psi}{ds}\left(\frac{1}{s} - 2s\right) - \frac{\psi}{4s^2} = 0.$$

by the standard transformation $\phi(s) = -2s\frac{d\ln\psi(s)}{ds}$. We then eliminate the linear term in this equation by writing $\psi = yv$ and then forcing the term linear in ψ' to be zero. This requirement gives the condition $\ln v' = s - 1/(2s)$, from which we find that the transformation $\phi(s) = 1 - 2s^2 - 2s\frac{d}{ds}\ln y(s)$ reduces the Ricatti equation (6.47) to a linear Schrödinger equation

$$\frac{d^2y}{ds^2} + (2 - s^2)y = 0. \tag{6.48}$$

Eq. (6.48) is the parabolic cylinder equation whose solution is a linear combination of the two linearly independent solutions, $y(s) = C_+D_{1/2}(s\sqrt{2}) + C_-D_{1/2}(-s\sqrt{2})$, with $D_\nu(x)$ the parabolic cylinder function of order ν. From the large-s behavior $\phi(s) \simeq (4s)^{-2}$, together with the asymptotics of $D_\nu(s)$, it follows that $C_- = 0$. Therefore the Laplace transform is

$$\phi(s) = 1 - 2s^2 - 2s\frac{d}{ds}\ln D_{1/2}(s\sqrt{2}). \tag{6.49}$$

The constant C_+ can be evaluated explicitly from the normalization condition $\phi'(0) = -C_+^{-1}$ and the properties[2] of $D_\nu(x)$. Using these facts, we find $C_+ = \Gamma(3/4)/\Gamma(1/4) = 0.337989\ldots$; this result should be compared with the exact value $C = (4\pi)^{-1/2} = 0.28209$.

The domain size distribution at large length can also be obtained from the small-s limit of the exact solution (6.49). The large-x tail of $\Phi(x)$ is exponential as follows from the behavior of the Laplace transform near its simple pole at $s = -\lambda$, $\phi(s) \simeq 2\lambda(s+\lambda)^{-1}$. The constant λ is given by the first zero of the parabolic cylinder function, $D_{1/2}(-\lambda\sqrt{2}) = 0$, located at $\lambda \approx 0.5409$. Therefore the domain size distribution asymptotically decays exponentially for large x

$$\Phi(x) \simeq A\exp(-\lambda x), \tag{6.50}$$

[2]The following properties are needed $D_\nu(0) = \frac{\sqrt{\pi}2^\nu}{\Gamma(1/2-\nu/2)}$, $D'_\nu(0) = \frac{\sqrt{\pi}2^{\nu+1}}{\Gamma(-\nu/2)}$, and $D_\nu(x) \sim x^\nu\exp(-x^2/4)[1 + \mathcal{O}(x^{-2})]$.

with amplitude $A = 2C\lambda$. The approximate value for the decay coefficient λ is larger than the exact value $\zeta(3/2)/\sqrt{16\pi} = 0.368468$.

While the independent interval approximation is not exact, it is very useful. By invoking the this approximation, we are able to write a closed master equation for the evolution of the domain size distribution. The independent interval approximation then yields the main qualitative behavior of the domain size distribution including: (i) the linear small-size limit of the distribution, (ii) the large-size exponential tail, and (iii) correct integrated properties, such as the $t^{-1/2}$ decay of the number of domains. As we shall see in later applications, the independent interval approximation applies to a wide range of coarsening processes.

6.3 Glauber Model in Greater Than One Dimension

Finite spatial dimension

When the spatial dimension is greater than one, the Ising model with Glauber kinetics is no longer solvable. Because of the importance of understanding coarsening phenomena in real systems, a variety of continuum models have been constructed that capture the essence of the Ising-Glauber model, and that are amenable to approximate analytical studies. These continuum descriptions will be the focus of the next chapter. However, within a description based on individual spins, there is still much that can be learned.

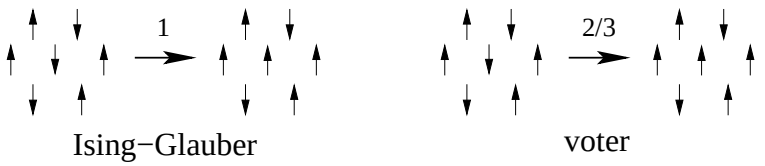

Figure 6.6: Comparison of the rates of an update event in the Ising-Glauber model at zero temperature and in the voter model on the triangular lattice.

First, we address the question about why the Ising-Glauber model is not soluble, while the closely related voter model is soluble in all dimensions. This dichotomy stems from a simple but profound difference between the Ising-Glauber model at zero temperature and the voter model. The zero-temperature limit is the appropriate situation to compare because once a domain of aligned spins (equivalently local consensus) is achieved in either model, there is no mechanism for a spin (or voter) in the interior of this domain to change its state. In one dimension, the Ising-Glauber and the voter model are identical because the three distinct types of transitions of energy lowering, energy neutral, and energy raising,

$$\downarrow\underline{\uparrow}\downarrow\to\downarrow\underline{\downarrow}\downarrow \qquad \uparrow\underline{\uparrow}\downarrow\to\uparrow\underline{\downarrow}\downarrow \qquad \uparrow\underline{\uparrow}\uparrow\to\uparrow\underline{\downarrow}\uparrow$$

respectively, occur with the same rates of 1, 1/2, and 0.

However, the two dynamics are different in a subtle but profound way in higher dimensions. In greater than one dimension, we again determine the transition rate for Glauber dynamics by using detailed balance (see also the discussion surrounding Eq. (6.51)). We thus obtain

$$\frac{w(s\to s_i')}{w(s_i'\to s)} = \frac{P(\{s_i'\})}{P(\{s\})} = \frac{e^{-\beta J s_i \sum s_j}}{e^{+\beta J s_i \sum s_j}} = \frac{1-\tanh(\beta J s_i \sum s_j)}{1+\tanh(\beta J s_i \sum s_j)} = \frac{1 - s_i \tanh \beta J \sum s_j}{1 + s_i \tanh \beta J \sum s_j}, \tag{6.51}$$

where the sum is over the nearest neighbors of s_i, and in the last step we used $\tanh(s_i x) = s_i \tanh x$ for $s_i = \pm 1$. Thus up to an overall constant that may be set to one, the transition for a given spin is

$$w(s_i) = \frac{1}{2}\left[1 - s_i \tanh\left(\beta J \sum_j s_j\right)\right]. \tag{6.52}$$

At zero temperature, this rule forbids energy raising updates, while energy lowering updates occur with rate 1 and energy conserving events occur with rate 1/2. Pictorially such an update is equivalent to *majority*

rule—a spin flips to agree with the majority of its neighbors (Fig. 6.6). In contrast, in the voter model, a particular voter can flip to the opposite state of its local majority with a probability equal to the fraction of neighbors in the minority state. That is, *proportional rule* dynamics. This proportionality of the update rule is the feature that makes it possible to factorize the master equation for the voter model into a product of one-dimension master equations that are then soluble in arbitrary spatial dimensions. There is no such simplification for the Ising-Glauber model. Moreover, when the spatial dimension is finite, the master equation is non-linear because s_j appears inside the hyperbolic tangent. For these reasons, most of our understand of the Ising-Glauber model in greater than one dimension is based on simulation results or on continuum theories, some of which will be discussed in the next chapter.

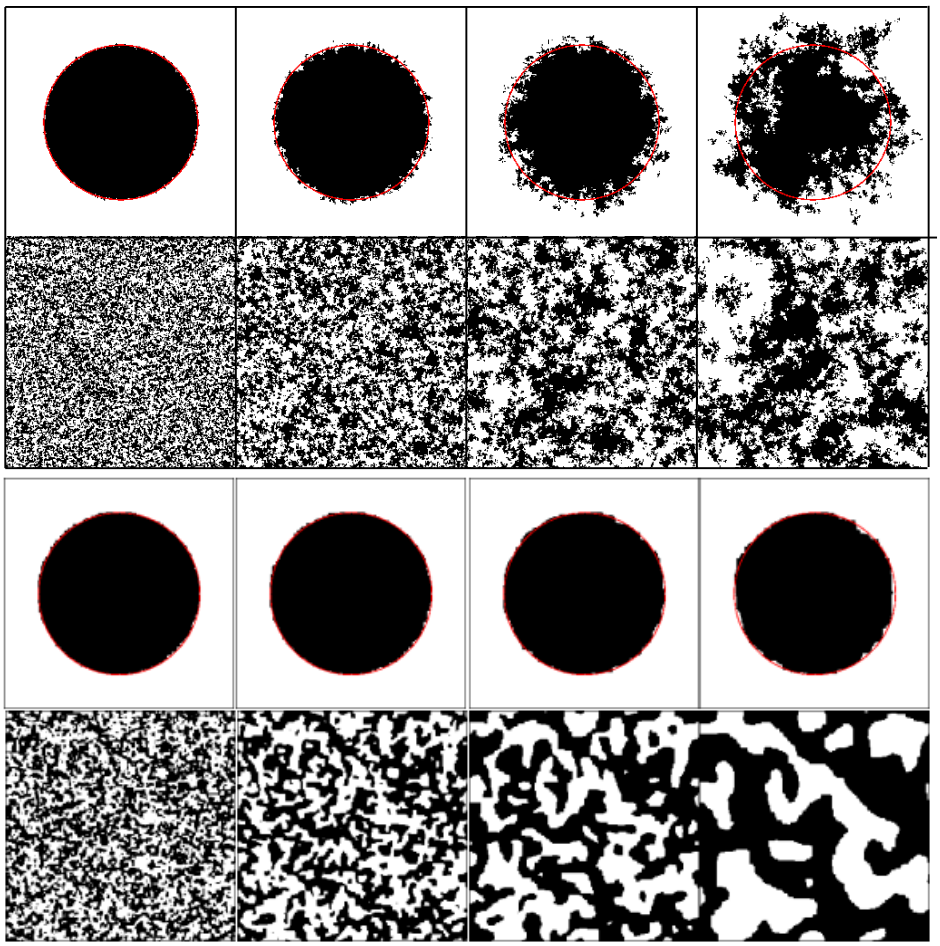

Figure 6.7: Spatial evolution in the voter model (top 2 rows) and the Ising-Glauber model at $T = 0$ (bottom two rows) on a 256×256 square lattice. Lines 1 & 3 shown snapshots at times $t = 4, 16, 64$, and 256 starting with an initial bubble of radius 180 for the voter model and the Ising-Glauber models, respectively. Lines 2 & 4 show the same evolution starting with a random initial condition with equal density of the two species. The voter model figure is from Dornic et al., Phys. Rev. Lett. **87**, 045701 (2001); courtesy of I. Dornic. The Ising-Glauber model figure is courtesy of V. Spirin.

Another important feature of proportional rule is that it leads to a process with little surface tension between domains of opposite-opinion voters. For example, a straight boundary between two opposite-opinion domains becomes fuzzier in voter model evolution (second line of Fig. 6.7). Additionally, even though the voter model undergoes coarsening, the weakness of surface tension means that the interface density disappears very slowly with time, namely, as $1/\ln t$. In contrast, for the Ising-Glauber model at zero temperature, there is a surface tension that scales as the inverse curvature. This fact is most easily shown by a continuum

approach that will be deferred until the next chapter. However, if we accept the existence of a surface tension that scales as the inverse curvature, then a single-phase droplet of radius R in a background of the other phase will shrink according to $\dot{R} \propto -1/R$, or $R(t)^2 = R(0)^2 - at$ and thus disappear in a finite time (third line of Fig. 6.7). Additionally, the surface tension will quickly eliminate high curvature regions so that the coarsening pattern is quite different from that of the voter model

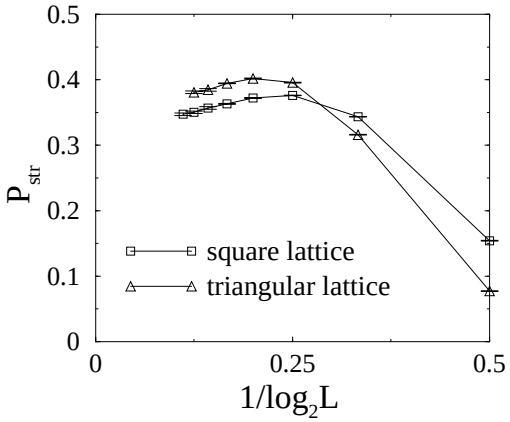

Figure 6.8: Probability that an $L \times L$ system (□ square lattice, Δ triangular lattice) eventually reaches a stripe state, $P_{\rm str}(L)$, as a function of $1/\log_2 L$ for L up to 512. Each data point, with error bars smaller than the size of the symbol, is based on $\geq 10^5$ initial spin configurations.

Perhaps the most basic questions about the Ising-Glauber model in greater than one dimension are concerned with the analog of the domain-size distribution. What is the nature of the coarsening when a system is prepared in a random initial state and then suddenly quenched to a low temperature? What is the final state? How long does it take to reach the final state? When the spatial dimension is greater than one and the temperature is below the critical temperature, the system organizes into a coarsening domain mosaic of up and down spins, with the characteristic length scale growing as $t^{1/2}$. Well-established continuum theories of spin dynamics with non-conserved order parameter show that the typical length scale of domains grows diffusively, namely, as $t^{1/2}$. For a finite system, this coarsening stops when the typical domain size reaches the linear dimension L of the system.

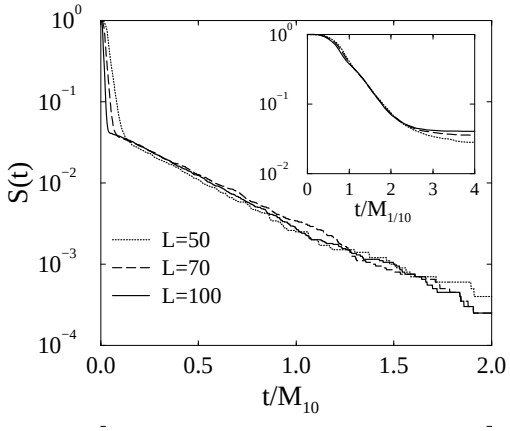

Figure 6.9: Time dependence of the survival probability $S(t)$ on $L \times L$ squares. Main graph: $S(t)$ versus t/M_{10} to highlight the long-time exponential tail. Here $M_k \equiv \langle t^k \rangle^{1/k}$ is the $k^{\rm th}$ reduced moment of the time to reach the final state. Scaling sets in after $S(t)$ has decayed to approximately 0.04. Inset: $S(t)$ versus $t/M_{1/10}$ to highlight the scaling and the faster exponential decay in the intermediate-time regime.

However, when the final temperature T of the quench is strictly zero, peculiar and unexpected anomalies

arise when the size of the system is finite. At early stages of the relaxation, there is little difference in the dynamics of $T = 0$ and $T > 0$ systems. However, when the elapsed time is such that the characteristic time of the coarsening is comparable to the time diffuse across the system, the two dynamics diverge. Perhaps the most striking feature of the $T = 0$ dynamics is that a system typically gets stuck in an infinitely long-lived metastable state. These metastable states consist of straight stripes in two dimensions while in higher dimension these metastable states are more complex and not so easily characterized. In two dimensions, the probability of getting stuck in a metastable state is approximately 1/3 as $L \to \infty$ (Fig. 6.8, while for $d \geq 3$ the ground state is essentially never reached.

Peculiar behavior is also exhibited by the survival probability $S(t)$ that the system has not yet reached its final state by time t. If the relaxation was purely diffusive with a characteristic time scale $\tau \sim L^2$, then the natural expectation is that the probability that a randomly-prepared system has not yet reach the final state, $S(t)$, would decay as $e^{-t/\tau}$. However, on the square lattice, $S(t)$ is controlled by two different time scales (Fig. 6.9). Initially, the characteristic time of the exponential decay scales as L^2, while at longer time this decay time grows as L^3.

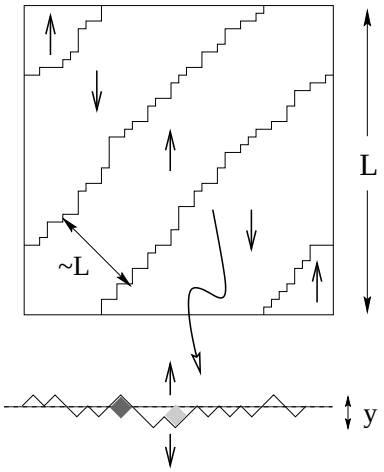

Figure 6.10: Diagonal stripe configuration on the square lattice with periodic boundaries. The lower portion shows part of one interface rotated by 45°. Zero-temperature Glauber dynamics is equivalent to particle deposition at the bottom of a valley (light-shaded square) – corresponding to the spin-flip event $\uparrow\to\downarrow$ – or particle evaporation from a peak (filled square) – corresponding to $\downarrow\to\uparrow$.

The source of the long-time anomaly in $S(t)$ arises from the approximately 4% of the configurations in which a diagonal stripe appears (Fig. 6.10). On the torus, this configuration consists of one stripe of $\uparrow$ spins and another of $\downarrow$ spins which, by symmetry, have width of order $L/2$. Each of these stripes winds once both toroidally and poloidally on the torus; they cannot evolve into straight stripes by a continuous deformation of the boundaries. Consequently a diagonal stripe configuration ultimately reaches the ground state.

Diagonal stripes are also extremely long-lived. To understand this long lifetime, we view a diagonal boundary as an evolving interface in a reference frame rotated by 45°. In this frame (Fig. 6.10 lower), a spin flip is equivalent either to "particle deposition" at the bottom of a valley ($\uparrow\to\downarrow$) or "evaporation" from a peak ($\downarrow\to\uparrow$). In a single time step each such event occurs with probability 1/2. For an interface with transverse dimension of order L, let us assume that there are of the order of L^μ such height extrema. Ref. **?** predicts $\mu = 1$, but we temporarily keep the value arbitrary for clarity. Accordingly, in a single time step, where all interface update attempts occur once on average, the interface center-of-mass moves a distance $\Delta y \sim L^{\mu/2}/L$ to give an interface diffusivity $D \sim (\Delta y)^2 \sim L^{\mu-2}$. We then estimate the lifetime $\tau_{\rm diag}$ of a diagonal stripe as the time for the interface to move a distance of order L to meet another interface. This gives $\tau_{\rm diag} \sim L^2/D \sim L^{4-\mu}$. Using the results of Ref. **?**, we expect $\tau_{\rm diag} \propto L^3$. The survival probability reflects these two time scales (Fig. 6.9).

In greater than two dimensions, the probability to reach the ground state rapidly vanishes as the system size increases. One obvious reason why the system "misses" the ground state is the rapid increase in the

number of metastable states with spatial dimension. This proliferation of metastable states makes it more likely that a typical configuration will eventually reach one of these states rather than the ground state.

Mean Field Theory

Mean field theory represents a special limit of a spin system in which fluctuations are negligible. One natural way to construct a mean-field description of a spin system is to replace the actual environment surrounding each spin by an average environment, that is then determined self consistently. Another simple way to achieve the mean-field limit is to embed the Ising model on a *complete graph* of N sites, where all the $N(N-1)/2$ pairs of spin interact with at the same strength. For this system, the Hamiltonian is

$$\mathcal{H} = -\frac{J}{N}\sum_{i<j} s_i s_j. \tag{6.53}$$

The coupling constant is chosen to scale inversely with the system size so that the energy is extensive, *i.e.*, scales linearly with N. The Ising model on a complete graph has the same equilibrium properties as the Curie-Weiss effective field theory.

By directly adapting the argument that gave the transition rate on a lattice in greater than one dimension (see Eq. (6.52)), the transition rate for Glauber dynamics on the complete graph is simply

$$w(s_i) = \frac{1}{2}\left[1 - s_i \tanh\left(\frac{\beta J}{N}\sum_j s_j\right)\right]. \tag{6.54}$$

where the sum $\sum s_j$ is over all other spins in the system. Now the equation of motion for the mean spin, or the one-point average, obeys $\frac{dS_i}{dt} = -2\langle s_i w_i\rangle$. Now we exploit the fact that there are no fluctuations in the magnetization to write $\langle f(m)\rangle = f(\langle m\rangle)$. With this identity we have $\langle\tanh\frac{\beta}{N}\sum_i s_i\rangle = \tanh\frac{\beta}{N}\sum_i\langle s_i\rangle = \tanh\beta m$, with $m = N^{-1}\sum_i\langle s_i\rangle$ the average magnetization. Thus the equation for the mean spin is

$$\frac{dS_i}{dt} = -S_i + \tanh\beta m. \tag{6.55}$$

Summing these rate equations, the average magnetization satisfies the rate equation

$$\frac{dm}{dt} = -m + \tanh\beta m. \tag{6.56}$$

In contrast to one dimension, the magnetization is generally not conserved. The rate equation has three fixed points, one at $m = 0$ and two at $\pm m_{\rm eq}$, with the latter determined by the roots of the familiar transcendental equation $m = \tanh(\beta J m)$. A linear stability analysis shows that the zero-magnetization state is stable for $\beta J \le 1$ but unstable for $\beta J > 1$, and vice versa for the state with $m = m_{\rm eq}$. Thus there is a phase transition at $\beta_c J = 1$. Near this critical point, the magnetization vanishes as $m_{\rm eq} \simeq [3(\beta - \beta_c)]^{1/2}$ or, as a function of the temperature,

$$m_{\rm eq} \sim (T_c - T)^{1/2} \tag{6.57}$$

as $T \to T_c$. The emergence of two equivalent, but symmetry-breaking ground states when the Hamiltonian is termed *spontaneous symmetry breaking*.

Landau Theory

Landau theory posits that the free energy $F(m)$: (i) is an *analytic* function of the order parameter (here, the magnetization m) and (ii) obeys the *symmetries* of the Hamiltonian (here, reflection symmetry $s \to -s$ and $F(m) = F(-m)$). For small magnetization, the magnetization is small and the free energy may be expanded as a Taylor series in even powers of m

$$F(m) = a_0 + a_2 m^2 + a_4 m^4 + \cdots . \tag{6.58}$$

Since we anticipate that there is an ordered low temperature phase and a disordered high temperature phase, the coefficient a_2 should change sign at the critical temperature T_c. The simplest assumption is that near the critical point, $a_2 \approx C(T - T_c)$. The Landau theory is equivalent to the Curie-Weiss effective field theory. With its remarkable simplicity, the Landau theory is powerful and applies widely. Nevertheless, this theory does not hold below the critical dimension, $d < d_c$, because the partition function and the free energy become non-analytic in the thermodynamic limit.

Moreover, we can extend the static mean-field theory to treat the time dependence of the magnetization. This approach will be discussed in more detail in the next chapter, but for the present discussion, we use the fact that the negative of the derivative of the free energy can be view as an effective force that drives the magnetization. Then the free energy that gives the equation of motion (6.56) from $dm/dt = -\delta F/\delta m$, is

$$F = C + \frac{1}{2}m^2 - \beta^{-1} \ln \cosh \beta m. \tag{6.59}$$

Expanding this free energy as a power series in the magnetization gives the Landau expansion $F(m) = C + \frac{1}{2}(\beta_c - \beta)m^2 + \frac{1}{12}\beta^3 m^4 + \cdots$. Below the critical temperature T_c, the free energy has two minima at $\pm m_{\rm eq}$ while for $T > T_c$ there is a single minimum at $m = 0$.

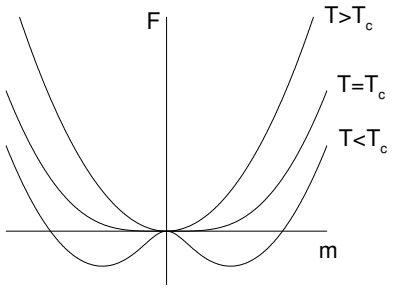

Figure 6.11: The free energy (6.59) versus the magnetization.

Above the critical temperature, the magnetization decays to zero and we expand $\tanh \beta m$ in Eq. (6.56) in powers of βm to give

$$\frac{dm}{dt} = -(\beta_c - \beta)m - \frac{1}{3}(\beta m)^3. \tag{6.60}$$

In the high temperature phase, the cubic term is negligible so that the magnetization decays exponentially in time, $m \sim \exp(-t/\tau)$ with $\tau = (\beta_c - \beta)^{-1}$. At the critical point, the relaxation becomes algebraic,

$$m \sim t^{-1/2} \tag{6.61}$$

in the long time limit. Below the critical temperature, the magnetization also decays exponentially toward its equilibrium value, $|m - m_{\rm eq}| \sim e^{-t/\tau}$, with $\tau^{-1} = 1 - \beta/\cosh^2(\beta m_{\rm eq})$. Thus, as the critical point is approached, either from above or from below, the relaxation time scale diverges as

$$\tau \sim |T_c - T|^{-1}. \tag{6.62}$$

The divergence of the relaxation time as $T \to T_c$ is a sign of the generic feature of critical slowing down where the approach to equilibrium becomes extremely slow.

6.4 Kawasaki Spin-Exchange Dynamics

The transition rate

As mentioned at the outset of this chapter, there are two fundamental classes of spin dynamics: magnetization conserving and magnetization non-conserving. The former class is appropriate to describe alloy systems, where the two different spin states naturally correspond to the two component atoms that comprise the alloy. In studying the dynamics of phase separation of an alloy into domains of pure metal, a plausible dynamics is that the positions of different species atoms are exchanged; there is no alchemy where one type of atom can be converted to the other type. In this section, we investigate a simple realization of order-parameter conserving dynamics that is also known as Kawasaki dynamics.

In Kawasaki dynamics, neighboring antiparallel spins simultaneously reverse their states so that

$$\cdots \underline{\uparrow\downarrow} \cdots \longrightarrow \cdots \underline{\downarrow\uparrow} \cdots . \tag{6.63}$$

Alternatively, the two spins can be regarded as being exchanged and hence the term *spin-exchange*. Clearly, such moves do not alter the magnetization. Thus the magnetization is *strictly* conserved in every update event. The existence of this strict conservation law has far-reaching consequences that will become more clear when we discuss continuum theories of spin dynamics in the next chapter.

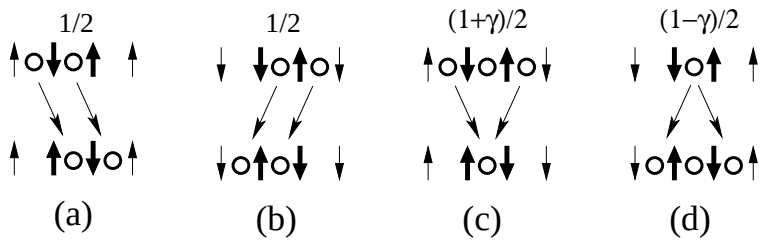

Figure 6.12: Energy neutral update events (a) & (b), energy lowering events (c), and energy raising events (d) for Kawasaki dynamics in one dimension. The spins that flip are shown bold. Also shown are the corresponding domain walls and the transition rates for these four events.

Again, there are three types of update events: energy raising, energy lowering, and energy neutral. As illustrated in Fig. 6.12, the energy neutral update is equivalent to the simultaneous hopping of two nearest-neighbor domain walls. As long as a bound domain-wall pair remains isolated from all other domain walls, the bound pair can hop freely between neighboring sites on the lattice. This pair can be viewed as an elementary excitation of the spin system. The diffusion rate of a domain wall pair merely sets the time scale, so there is no loss of generality in setting this rate to 1/2, as in Glauber dynamics. Because such diffusive moves do not alter the energy, they automatically satisfy the detailed balance condition. The rates of the remaining two update events are then set by detailed balance. Since spin exchange involves the interactions among four spins—the two spins that flip and their two neighbors—the rates depend on the total energy of the three bonds connecting these four spins. The detailed balance condition is

$$\frac{w_3}{w_{-1}} = \frac{p_{-1}}{p_3} = \exp(4\beta J), \tag{6.64}$$

where w_q is the transition rate out of a state with energy qJ and p_q its equilibrium probability. Using the convenient Glauber notations of $w_3 = (1+\gamma)/2$ and $w_{-1} = (1-\gamma)/2$ for energy raising and energy lowering transitions, the detailed balance condition has the the same form as in Glauber dynamics, $\frac{1+\gamma}{1-\gamma} = \exp(4\beta J)$. Therefore $\gamma = \tanh 2\beta J$, just as in Glauber dynamics.

To determine the formal expression for the transition rates, we first must guarantee that spins i and $i+1$ are antiparallel. This constraint can be achieved by the factor $(1 - s_i s_{i+1})/2$ that equals $+1$ if the two spins are antiparallel and equals zero otherwise. The inner workings of the flip rate then depends on the interaction energy between the pairs s_{i-1} and s_i, and between s_{i+1} and s_{i+2}. The flip rate should be a symmetric function of these two bond energies and the rate should be proportional to $(1+\gamma)/2$, $1/2$, and

$(1-\gamma)/2$, when the signs of these bond energies are $--$, $+-$, and $++$ respectively. These constraints impose the following form for the transition rate:

$$w_i(s_i, s_{i+1}) = \frac{1}{2}\left[1 - \frac{\gamma}{2}(s_{i-1}s_i + s_{i+1}s_{i+2})\right] \times \frac{1}{2}(1 - s_i s_{i+1}). \tag{6.65}$$

An important feature of this transition rate is that the evolution of spin correlation functions are no longer closed. One-spin averages are coupled to three-spin averages, two-spin averages are coupled to four-spin averages, *etc.* Thus the equation of motion for a particular correlation function generates an infinite hierarchy of equations for high-order correlations. This coupling to higher-order correlation functions arises in a wide range of many-body problems and a crucial art is find a tractable and accurate scheme to truncate this infinite hierarchy.

Frustration at zero temperature

In this section, we investigate the evolution of a one-dimensional system by Kawasaki dynamics at zero temperature. Because Kawasaki dynamics is more constrained than Glauber dynamics, a system will almost always get stuck in one of the very large number of metastable states; a similar phenomenon occurs in higher dimensions. In the context of the present discussion, a metastable state is one whose energy is above the ground state energy and for which the only possible transitions by Kawasaki dynamics would raise the energy (see Fig. 6.12(d)). At zero temperature such transitions cannot occur so that the system is stuck forever in a metastable state in which each domain particle is separated by more than a nearest-neighbor distance from any other domain particle. Equivalently the lengths of all spin domains are two or longer. The number of such configurations in a system of length L asymptotically grows as g^L, where $g = (1+\sqrt{5})/2$ is the golden ratio. It is striking how often this beautiful number appears in statistical physics problems. At zero temperature the multitude of frustrated states is an obstacle that prevents the system from reaching the ground state. At non-zero temperature, these states merely slow the approach toward equilibrium.

To study how the system evolves to a metastable state, we study the case where energy lowering transitions only are allowed, as illustrated in Fig. 6.12(a)–(c)). The resulting behavior differs only slightly from the situation where diffusive moves are also allowed, but the former case is much simpler to treat analytically. The dynamics is perhaps best visualized in terms of the domain walls that occupy the sites of the dual lattice. According to Fig. 6.12(c), an update step consists of picking three contiguous domain wall particles at random and then removing the two side particles. Since pairs of domain walls are removed sequentially from triplets of consecutive domain walls, the process is equivalent to the random sequential adsorption of $\cdot\circ\cdot$ "forks" wherever a string of three consecutive domain wall particles exists. Because of this equivalence, we can use the tools of random sequential adsorption (Chapter 5) to solve the problem.

Let E_k be the probability that a string of k sites (in the dual lattice) are all occupied by domain walls. This probability evolves by the master equation

$$\frac{dE_k}{dt} = -(k-2)E_k - 2E_{k+1} - 2E_{k+2} \tag{6.66}$$

for $k \geq 3$. This equation reflects the different ways that the transition $\circ\circ\circ \to \cdot\circ\cdot$ can occur and alter the number of empty strings of length k. There are $k-2$ ways that this transition can occur in the interior of a k-string. There are also 2 ways that this transition can occur with two sites at the edge of the k-string and one site outside, and also 2 ways with one site at the edge of the k-string and two sites outside.

We solve this rate equation by introducing the exponential ansatz $E_k = \phi(t)\exp[-(k-2)t]$ (see also the discussion surrounding Eq. (5.3)). For the initial condition of an antiferromagnetic spin state, the dual lattice is completely occupied. Thus $E_k = 1$ initially, so that $\phi(0) = 1$. Substituting this ansatz into the rate equation (6.66) leads to the ordinary differential equation

$$\frac{d\phi}{dt} = -2\phi(e^{-t} + e^{-2t}). \tag{6.67}$$

Integrating this equation gives the string probabilities for $k \geq 2$,

$$E_k(t) = \exp\left[-(k-2)t + e^{-2t} + 2e^{-t} - 3\right]. \tag{6.68}$$

Since two domain walls are lost in each update event and these events occur with rate E_3, the domain wall density $\rho \equiv E_1$ satisfies $\frac{d\rho}{dt} = -2E_3$. Using Eq. (6.68) for E_3 and integrating then yields the domain wall density

$$\rho(t) = 1 - 2\int_0^t ds \exp\left[-s + e^{-2s} + 2e^{-s} - 3\right]. \tag{6.69}$$

The final "jamming" density is finite, $\rho_{\rm jam} \equiv \rho(\infty) = 0.450898\ldots$. Thus there is not very much relaxation as almost half of the domain walls still remain in the final jammed state. Moreover, the relaxation to the jamming density is exponential in time,

$$\rho(t) - \rho_{\rm jam} \simeq e^{-3}e^{-t}. \tag{6.70}$$

We see that the system neither reaches the lowest energy state, nor does it exhibit critical slowing down. The underlying reason for both of these unphysical behaviors is that the dynamics samples only a very restricted portion of the phase space.

Coarsening at infinitesimal temperature

While the one-dimensional chain with Kawasaki dynamics quickly reaches a jammed state when the temperature is zero, the ground state will be reached for any non-zero temperature, no matter how small. At very low temperatures, energy raising updates will occur, albeit very rarely, so that the system is able to sample all of the phase space and come to equilibrium. Because the correlation length diverges as the temperature approaches zero, one can set the temperature sufficiently small so that the correlation length is much larger than the length of the system. Consequently, the entire system belongs to one domain and the equilibrium state of the system is effectively the same as the ground state.

The large separation of time scales between energy raising updates and all other update events leads to an appealing description of the domain evolution within the framework of an extremal dynamics. Since the rate of an energy raising update equals $e^{-4\beta J}$, the typical time for such an event is $\tau \equiv e^{4\beta J}$. We define a time scale such that $e^{4\beta J}$ represents the time unit. Energy neutral and energy lowering events then occur instantaneously in this time unit. Starting from an initial state, the system instantly reaches a frustrated state in which no further energy neutral or energy lowering moves are possible. After a time τ has elapsed (on average) an energy raising event occurs that is then followed by a burst of energy neutral and energy lowering events until the system reaches another frustrated state. This pattern of an energy raising event followed by a burst of complementary events continues until a finite system reaches the ground state. As we will now show, this dynamics leads to the typical domain size growing in time as $t^{1/3}$. This growth law is a general feature of order-parameter conserving dynamics. One of the appealing features of Kawasaki dynamics in one dimension is that this important result of $t^{1/3}$ coarsening emerges from a direct and unambiguous calculation. In contrast, we will see in the next chapter that it is much more subtle to deduce the $t^{1/3}$ scaling law from continuum approaches.

At long times, the system will evolve to a low-energy state that consists of alternating domains of typical length ℓ. The evolution in the low-temperature limit is controlled by rare, energy raising updates where a pair of domain walls nucleates around an existing isolated domain wall. Once this triplet of domain walls forms, a bound pair of these domain walls can diffuse freely with no energy cost until another isolated domain wall is encountered. When such a collision occurs, two of the domain walls annihilate so that a static single domain wall remains. As illustrated in Fig. 6.13, the creation of a mobile bound domain wall pair is equivalent to an isolated spin splitting off from a domain and then diffusing freely within a neighboring domain of length ℓ of the opposite orientation. If this diffusing spin returns to its starting point, the net effect is no change in the domain configuration. However, if the spin manages to traverse to the other side of the domain, then one domain has increased its size by one and another has shrunk by one. This effective diffusion of domain lengths is the mechanism that drives the coarsening of the system.

What is the probability that the spin can actually traverse to the other side of the domain? This is a classic first-passage probability problem (see the highlight on the next page). Once the spin has split off, it is a distance one from its initial domain and a distance $\ell - 1$ from the domain on the other side. Since the spin diffuses freely, with probability $1/\ell$ it eventually reaches the other side, while with probability $1 - 1/\ell$

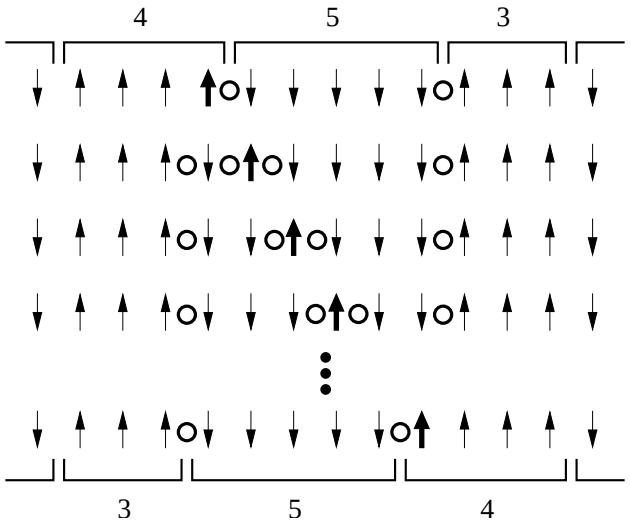

Figure 6.13: Illustration of the effective domain diffusion from Kawasaki dynamics at infinitesimal temperature. The second line shows an energy raising event where a spin (shown bold) splits off from a domain. Eventually this spin joins the next domain to the right. Also shown is the evolution of the domain walls. The net result of the diffusion of the spin across the middle domain is that this moves one step to the left.

the spin returns to its starting position. Thus the probability that the ℓ-domain hops by one step equals $1/\ell$. That is, the diffusion coefficient of a domain equals the inverse of its length.

$$D(\ell) = \ell^{-1}. \tag{6.71}$$

First passage probability and the gambler's ruin problem

Consider a random walk in a finite interval of length N. The two boundary sites are absorbing, *i.e.*, the random walker immediately disappears upon reaching these sites. Suppose that the starting position of the random walk is n, with $0 \le n \le N$. What is F_n, the probability that the walker first reaches the boundary at site N? We can write a simple recursion formula for the first-passage probability. With probability $1/2$, the walk steps to site $n-1$, at which point the exit probability to site N is F_{n-1}. Similarly, the walk steps to site $n+1$ with probability $1/2$, where the exit probability is F_{n+1}. Thus the first passage probability satisfies the discrete Poisson equation

$$F_n = \frac{1}{2}(F_{n-1} + F_{n+1}), \tag{6.72}$$

with the boundary conditions $F_0 = 0$ and $F_N = 1$. The solution is simple:

$$F_n = \frac{n}{N}. \tag{6.73}$$

This first passage probability also solves a neat probability theory problem. In a fair coin-toss game, the probability that a gambler ruins a Casino equals the wealth of the gambler divided by the combined wealth of the gambler and casino. Gambling is most definitely a bad idea...

Thus in the low-temperature limit, the spin dynamics translates to an effective isotropic hopping of entire domains by one step to the left or to the right[3] (Fig. 6.14). Domains of length one disappear whenever one of their neighboring domain hops toward them. Concomitantly, the lengths of the neighboring domains are rearranged so that four domains merge into two (Fig. 6.15).

The net effect of domain hopping is domain coarsening because domains of length one disappear whenever one of their neighbors hops toward them. We now determine the typical domain length from the following

[3]There is an anomaly involving domains of length 2 that can be ignored for the purposes of this discussion.

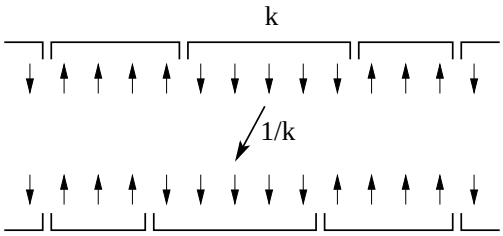

Figure 6.14: Effective domain diffusion from Kawasaki dynamics at infinitesimal temperature.

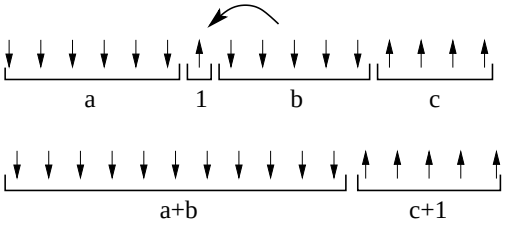

Figure 6.15: The outcome after domain merging.

heuristic argument. Because each domain performs a random walk, coalescence occurs whenever a domain diffuses of the order of its own length. In a coalescence, a domain typically grows by an amount $\Delta\ell$ that is also of the order of ℓ. The time between coalescence events is $\Delta t \sim \ell^2/D(\ell)$. We then have

$$\frac{\Delta\ell}{\Delta t} \sim \frac{\ell}{\ell^2/D(\ell)} \sim \frac{1}{\ell^2}.$$

Thus domain growth is subdiffusive

$$\ell \sim t^{1/3}. \tag{6.74}$$

It is conventional to define the *dynamical exponent* z in terms of the growth of the typical length scale in a coarsening process via $\ell \sim t^z$. For the non-conserved Glauber and the conserved Kawasaki dynamics, the dynamical exponent is:

$$z = \begin{cases} 1/2 & \text{nonconservative dynamics,} \\ 1/3 & \text{conservative dynamics.} \end{cases} \tag{6.75}$$

While we have derived these results in one dimension, they are generic for all spatial dimensions. Conservation laws are a crucially important ingredient in determining the nature of non-equilibrium dynamics.

6.5 Cluster Dynamics

Glauber single-spin flip dynamics and the Kawasaki spin-exchange dynamics are local in that they involve flipping a single spin or a pair of spins. Because of their simplicity and their plausibility in describing the evolution of real systems, these rules were the basis of many simulational studies of coarsening and dynamic critical phenomena. However, a dynamics that is based on flipping a small number of spins is perforce not computationally efficient. Compounding this inefficiency, the dynamics significantly slows down close to criticality.

To mitigate the effects of critical slowing down, Swendsen and Wang developed a clever dynamical update rule in which an entire suitably-defined cluster of spins is flipped simultaneously. A crucial feature of this dynamics is that it significantly reduces the effect of critical slowing down because clusters become large near the critical point, so that a large number of spins is flipped in a single update near criticality. Because of this computational efficiency, such cluster algorithms have been used extensively to elucidate the equilibrium behavior of many-body statistical mechanical and lattice field theory models. The Swendsen-Wang and the

Wolff algorithms are two of the earliest and most prominent such examples of this type of cluster dynamics. In this section, we analyze the domain size distribution in one dimension at zero temperature for these two dynamical rules.

For local non-conservative dynamics, such as the Glauber model, the time T_L required for a system of length L to reach the ground state increases as L^2 because of the underlying diffusive nature of the dynamics. In contrast, for the Swendsen-Wang cluster algorithm, this time to reach the ground state is much smaller, and grows only logarithmically with the system size, $T_L \sim \ln L$. The Wolff algorithm is even more dramatic; the time to reach the ground state remains *finite* even as the length of the system diverges. Remarkably, the domain length distribution can be obtained analytically for both Swendsen-Wang and Wolff dynamics, even though the solution remains elusive when spins are flipped individually according to Glauber dynamics.

Swendsen-Wang dynamics

In one dimension, an Ising spin chain consists of alternating spin-up and spin-down domains. In the Swendsen-Wang algorithm, an entire domain of aligned spins is chosen at random and all these spins are flipped simultaneously as illustrated below:

$$\cdots\uparrow\underbrace{\downarrow\downarrow\downarrow\downarrow\downarrow}\uparrow\cdots\longrightarrow\cdots\uparrow\underbrace{\uparrow\uparrow\uparrow\uparrow\uparrow}\uparrow\cdots.$$

By construction, all such update events decrease the energy. We also take the flip rate to be one without loss of generality. We now present a master equation solution of this dynamics.

In each update event, there is a net loss of two domains. Since the flip rate is unity, the number density of domains ρ decreases according to

$$\frac{d\rho}{dt} = -2\rho. \tag{6.76}$$

Hence the density of domains decreases exponentially with time, and for the antiferromagnetic initial condition in which $\rho(0) = 1$, the domain density is

$$\rho(t) = e^{-2t}. \tag{6.77}$$

The average domain length is the inverse of the domain density and thus grows exponentially with time, $\langle k\rangle = e^{2t}$. When this average length reaches the system length L the dynamics is complete. This criterion yields the time to reach the ground state $T_L \propto \ln L$.

Now let's turn to the domain length distribution. We define c_ℓ as the density of domains of length ℓ. Domains undergo a three-body aggregation process: when a domain is flipped, it merges with its two neighbors. The length of the resulting domain equals the length of the three constituent domains. Therefore c_ℓ evolves according to

$$\frac{dc_\ell}{dt} = -3c_\ell + \frac{1}{\rho^2}\sum_{i+j+k=\ell} c_i\, c_j\, c_k. \tag{6.78}$$

The factor of $-3c_\ell$ accounts for the loss that occurs when a domain of length ℓ or either of its neighboring domains is flipped. The last term accounts for the gain in c_ℓ due to the flipping of a domain of length j that then merges with its two neighboring domains of lengths i and k, with the constraint that $\ell = i + j + k$. The simplest way to deduce the prefactor ρ^{-2} is to make this master equation consistent with Eq. (6.76). Alternatively, the convolution terms are products of the length density of the flipped domain times the normalized length densities of its two neighbors. Newly created domains do not affect their neighbors, nor are they affected by their neighbors. Thus if the domains are initially uncorrelated, they remain uncorrelated. Because no spatial correlations are generated, the rate equations are exact!

We can obtain a cleaner-looking master equation by introducing $P_\ell \equiv c_\ell/\rho$, namely, the probability for a domain of length ℓ (with the normalization $\sum_\ell P_\ell = 1$). Using Eqs. (6.76) and (6.78), P_ℓ evolves according to

$$\frac{dP_\ell}{dt} = -P_\ell + \sum_{i+j+k=\ell} P_i P_j P_k. \tag{6.79}$$

As we have seen previously, the convolution form of the gain term cries out for applying the generating function method. Thus we introduce the generating function $F(z) = \sum_\ell P_\ell\, z^\ell$ into (6.79) and find that it

satisfies $\frac{\partial F}{\partial t} = -F + F^3$. We then solve this differential equation by writing $1/(F^3 - F)$ in a partial fraction expansion so that the resulting equation can be integrated by elementary methods. The solution is

$$F(z,t) = \frac{F_0(z)e^{-t}}{\sqrt{1 - F_0(z)^2(1 - e^{-2t})}}, \tag{6.80}$$

where $F_0(z)$ is the initial generating function.

For the antiferromagnetic initial condition, the initial condition is $F_0(z) = z$. Expanding the generating function in powers of z then yields the domain number distribution

$$P_{2\ell+1} = \binom{2\ell}{\ell} \left(\frac{1 - e^{-2t}}{4}\right)^{\ell} e^{-t} \tag{6.81}$$

in which domains have odd lengths only. Since the average domain length grows exponentially with time, $\langle \ell \rangle = e^{2t}$, we expect that this scale characterizes the entire length distribution. Employing Stirling's approximation, we find that asymptotically the length distribution approaches the scaling form $P_\ell \to e^{-2t}\Phi(\ell e^{-2t})$ with the scaling function

$$\Phi(x) = \frac{1}{\sqrt{2\pi x}}\, e^{-x/2}. \tag{6.82}$$

Because the scaling function diverges $\Phi(x) \sim x^{-1/2}$ for $x \ll 1$, there is a large number of domains whose length is smaller than the average. As in the example of constant kernel aggregation, domains whose length is larger than the average are also exponentially rare, although here there is an additional weak algebraic correction.

It is a fun exercise to extend the zero-temperature Swendsen-Wang dynamics, with energy lowering moves only, to the $q = \infty$ Potts model. Now a domain merges with only one of its neighbors. Since there is a net loss of one domain in a single update, the number density obeys $d\rho/dt = -\rho$. Therefore $\rho(t) = e^{-t}$ while the average domain length again grows exponentially with time. The domain length size distribution now evolves by two-body aggregation, so that this distribution satisfies

$$\frac{dP_k}{dt} = -P_k + \sum_{i+j=k} P_i P_j. \tag{6.83}$$

To solve this equation, we again introduce the generating function into this equation to give $\frac{\partial F}{\partial t} = -F + F^2$, whose solution is simply

$$F(z,t) = \frac{F_0(z)e^{-t}}{1 - (1 - e^{-t})F_0(z)}. \tag{6.84}$$

Expanding this generating function in a power series we immediately obtain

$$P_k(t) = e^{-t}(1 - e^{-t})^{k-1}. \tag{6.85}$$

Asymptotically, the distribution attains the scaling form $P_k(t) \sim e^{-t}\Phi(ke^{-t})$ with the purely exponential scaling function $\Phi(x) = \exp(-x)$. The enhancement of smaller than average domains disappears in the $q \to \infty$ limit.

Wolff dynamics

In the Wolff cluster algorithm, a spin is selected at random and the domain it belongs to is flipped. This protocol further accelerates the dynamics compared to the Swendsen-Wang algorithm because the larger the domain, the more likely it is to be updated. Schematically, the Wolff dynamics is

$$\cdots \uparrow \underbrace{\downarrow\downarrow \cdots \downarrow\downarrow}_{k} \uparrow \cdots \xrightarrow{k} \cdots \uparrow \underbrace{\uparrow\uparrow \cdots \uparrow\uparrow}_{k} \uparrow \cdots, \tag{6.86}$$

so that a flipped domain again simply merges with its neighbors. Since each spin is selected randomly, the time increment associated with update is the same. The total domain density therefore decreases with constant rate

$$\frac{d\rho}{dt} = -2. \tag{6.87}$$

As a result, the domain density decreases linearly with time, $\rho(t) = 1-2t$ and the entire system is transformed into a single domain in a finite time, $t_c = 1/2$. Correspondingly, the average domain length, $\langle k\rangle = (1-2t)^{-1}$, diverges as $t \to t_c$.

The evolution of the domain length distribution is governed by the natural generalization of (6.79)

$$\frac{dP_\ell}{dt} = -\ell P_\ell + \sum_{i+j+k=\ell} jP_iP_jP_k. \tag{6.88}$$

The generating function $F(z,t) = \sum_\ell P_\ell z^\ell$ satisfies

$$\frac{\partial F}{\partial t} = z\,(F^2-1)\,\frac{\partial F}{\partial z}. \tag{6.89}$$

To solve this equation, we first transform the variables from (t,z) to $(\tau,y) \equiv (t, t-\ln z)$ to absorb the negative term on the right-hand side. This transformation leads to

$$\frac{\partial F}{\partial \tau} = -F^2\,\frac{\partial F}{\partial y}. \tag{6.90}$$

We now employ the same procedure as that used in the solution of aggregation with the product kernel (see the discussion of Eqs. (3.23)–(3.25) in chapter 3) to transform among the variables (τ, y, F) and reduce (6.90) into the linear differential equation $\frac{\partial y}{\partial \tau} = F^2$. The solution to this equation is simply $y = G(F) + F^2\tau$, with $G(F)$ determined by the initial conditions, or, equivalently

$$t - \ln z = G(F) + F^2 t. \tag{6.91}$$

For the antiferromagnetic initial condition $F_0(z) = z$, so that $G(F) = -\ln F$. Substituting $G(F) = -\ln F$ into (6.91) and exponentiating yields the following implicit equality satisfied by the generating function

$$z = F\,e^{t-F^2t}\,. \tag{6.92}$$

The length distribution P_k is just the k^{th} term in the power series expansion of $F(z)$. Formally, this term may be extracted by writing P_k in terms of the contour integral

$$P_k = \frac{1}{2\pi i}\oint \frac{F(z)}{z^{k+1}}\,dz\,,$$

and then transforming the integration variable from z to F (see the discussion of the Lagrange inversion formula in Chapter 3). This procedure gives

$$\begin{aligned} P_k = \frac{1}{2\pi i}\oint \frac{F(z)}{z^{k+1}}\,dz &= \frac{1}{2\pi i}\oint \frac{F}{z(F)^{k+1}}\,\frac{dz}{dF}\,dF, \\ &= \frac{e^{-kt}}{2\pi i}\oint e^{kF^2t}\left[\frac{1}{F^k} - \frac{2t}{F^{k-2}}\right]\,dF, \end{aligned} \tag{6.93}$$

where we use the fact that $\frac{dz}{dF} = e^{t-F^2t}(1-2F^2t)$ in the above integral. Now we find the residues simply by expanding e^{kF^2t} in a power series and keeping only the coefficient of $\frac{1}{F}$ in the integrand. Because the power series is even in F, only P_k for odd values of k is non zero. This procedure gives:

$$P_k = e^{-kt}\left[\frac{(kt)^{(k-1)/2}}{\left(\frac{k-1}{2}\right)!} - 2t\frac{(kt)^{(k-3)/2}}{\left(\frac{k-3}{2}\right)!}\right].$$

After some simple algebra, the domain length distribution is

$$P_{2k+1}(t) = \frac{(2k+1)^{k-1}}{k!}\,t^k\,\exp[-(2k+1)t]\,. \tag{6.94}$$

Near the critical time ($t \to 1/2$), Stirling's approximation gives, for the leading behavior of domain length distribution

$$P_k(t) \simeq \frac{1}{\pi^{1/2}}\,\frac{1}{k^{3/2}}\,e^{\left[-k\,\rho^2/4\right]}\,, \tag{6.95}$$

with $\rho = 1 - 2t$. It would be natural to expect that the distribution has the characteristic length scale $k_* = 4c^{-2} = (1/2 - t)^{-2}$. However, this length does not fully characterize the distribution. While most domains are short near the critical time, because the domain length distribution has an algebraic $k^{-3/2}$ tail that is cutoff at a value $k^* \propto \rho^{-2}$, the average domain length diverges as ρ^{-1}.

There is a similarity between coarsening by the Wolff dynamics and the gelation transition in product-kernel aggregation. In both cases, a giant component emerges in a finite time. For the Wolff dynamics, the gelation transition is discontinuous; the gel mass is zero prior to the transition and it becomes one immediately after the transition. Thus gelation in the Wolff dynamics is a first-order (discontinuous) phase transition. In contrast, for product-kernel aggregation, gelation is a second-order (continuous) transition.

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Problems

Problems

1. Evaluate the average opinion for a Democrat in a sea of uncommitted voters: $S(\mathbf{x}, 0) = \delta(\mathbf{x})$.

4.1 Glauber Spin-Flip Dynamics

1. Solve for the domain number distribution in the Potts model with Glauber dynamics. Hint: P_1 is replaced by $P_1/(q-1)$ in Eq. (6.45).
2. verify that for the Hamiltonian $\mathcal{H} = -\sum_{i<j} J_{i,j} s_i s_j$ the spin flip rate is $w_i = \frac{1}{2}\big(1 - \tanh \beta s_i \sum_j J_{i,j} s_j\big)$.
3. Obtain the entropy in the mean-field model
4. Determine $P_M(t)$, the probability to have M up spins and $N - M$ down spins,[4] for the zero temperature dynamics.
5. Examine $P_M(t)$ for the critical dynamics.
6. In the low temperature regime ($\infty > \beta > \beta_c$), the distribution $P_M(t)$ is bimodal with peaks of width $\propto \sqrt{N}$ around $M_\pm = \frac{1}{2} N (1 \pm m_\infty)$. The system spends almost all time in the proximity of the peaks yet occasionally it leaves one peak and reaches the other. Estimate the transition time.

4.2 Glauber spin-exchange dynamics

1. Show that the correlation functions obey an infinite hierarchy of equations. Write the evolution equation for S_i.
2. Obtain the number of frozen configurations in the zero-temperature for Kawasaki dynamics.
3. Solve for the domain wall density at zero-temperature for random initial conditions.

4.2 Extremal dynamics

1. Analyze a domain coarsening process where the smallest domain merges with one of its neighbors.

4.5 Asymmetric exclusion process

1. Solve for the steady state in a closed system with random addition and deletion of particle at each lattice site.

[4] In all problems in this paragraph the system is *finite*.