Appendix A

MATTERS OF TECHNIQUE

A.1 Relation between Laplace Transforms and Real Time Quantities

In many time-dependent phenomena, we want the asymptotic behavior of some quantity as a function of time when only its generating function or its Laplace transform is available. A typical example is a function $F(t)$ whose Laplace transform has the small-s behavior $F(s) \sim s^{\mu-1}$, or equivalently, whose generating function has the form $F(z) \sim (1-z)^{\mu-1}$, with $\mu < 1$ as $z \to 1$ from below. We will show that the corresponding time dependence is $F(t) \sim t^{-\mu}$ as $t \to \infty$. The first point to make is that the generating function and the Laplace transform are equivalent. For a function $F(t)$ that is defined only for positive integer values, that is $t = n$, with n an integer, the Laplace transform is

$$F(s) = \int_0^\infty F(t)\, e^{-st}\, dt = \sum_{t=0}^{\infty} F(n)\, e^{-sn}.$$

Thus defining $z = e^{-s}$, the Laplace transform is the same as the generating function. In the limit $z \to 1$ from below, the sum cuts of only very slowly and can be replaced by an integral. This step leads to the Laplace transform in the limit $s \to 0$.

If we are interested only in long-time properties of a function $F(t)$, these features can be obtained through simple and intuitively appealing means. While lacking in rigor, this approach provides the correct behavior for all cases of physical interest. The most useful of these methods are outlined in this section. Let us first determine the long-time behavior of a function $F(t)$ when only its Laplace transform is known. There are two fundamentally different cases to consider: (a) $\int_0^\infty F(t)\, dt$ diverges or (b) $\int_t 0^\infty F(t)\, dt$ converges.

In the former case, relate the Laplace transform $F(s)$ to $F(t)$ by the following simple step:

$$F(s) = \int_0^\infty F(s)\, e^{-st}\, dt \approx \int^{t^*} F(t)\, dt. \tag{A.1}$$

That is, we simply replace the exponential cutoff in the integral, with characteristic lifetime $t^* = 1/s$, by a step function at t^*. Although this crude approximation introduces numerical errors of the order of 1, the essential asymptotic behavior of $F(t)$ is preserved when $\int 0^\infty F(t)\, dt$ diverges. Now if $F(t) \to t^{-\mu}$ with $\mu < 1$ as $t \to \infty$, then $F(s)$ diverges as

$$F(s) \sim \int^{1/s} t^{-\mu}\, dt \sim s^{\mu-1} \tag{A.2}$$

as $s \to 0$ from below. In summary, the fundamental connection when $\int_0^\infty F(t)\, dt$ diverges is

$$F(t) \sim t^{-\mu} \quad \longleftrightarrow \quad F(s) \sim s^{\mu-1}. \tag{A.3}$$

The above result also provides a general connection between the time integral of a function, $\mathcal{F}(t) \equiv \int_0^t F(t)\, dt$, and the Laplace transform of F. For $s = 1/t^*$ with $t^* \to \infty$, Eq. (A.2) is just the following statement:

$$F(s = 1/t^*) \sim \int_0^{t^*} F(t)\, dt = \mathcal{F}(t^*). \tag{A.4}$$

Thus a mere variable substitution provides an approximate, but asymptotically correct, algebraic relation between the Laplace transform of a function and the time integral of this same function. For this class of examples, there is no need to perform an integral to relate a function and its Laplace transform.

Conversely, when $\int_0^\infty F(t)\, dt = \mathcal{F}(\infty)$ converges, we can obtain the connection between $F(t)$ and $F(s)$ in a slightly different way. Let us again suppose that $F(t) \sim t^{-\mu}$ as $t \to \infty$, but now with $\mu > 1$ so that $F(s)$ is finite as $s \to 0$. Exploiting the fact that the time integral of $F(t)$ converges, we write

$$\begin{aligned}
F(s) &= \int^\infty t^{-\mu} \left[1 - (1 - e^{-st})\right]\, dt, \\
&\sim \mathcal{F}(\infty) + \int_{1/s}^\infty t^{-\mu}\, dt, \\
&\sim \mathcal{F}(\infty) + s^{\mu - 1}.
\end{aligned} \tag{A.5}$$

Again, we replace the exponential cutoff in the integrand by a sharp cutoff.

In summary, the small-s behavior of the Laplace transform, or, equivalently, the $z \to 1$ behavior of the generating function, are sufficient to determine the long-time behavior of the function itself. Since the transformed quantities are usually easy to obtain by the solution of an appropriate boundary-value problem, the asymptotic methods outlined here provide a simple route to obtain long-time behavior.

In the context of time-dependent phenomena, one of the most useful features of the Laplace transform is that it encodes all positive integer powers of the mean time. That is, we define the positive integer moments of $F(t)$ as

$$\langle t^n \rangle = \frac{\int_0^\infty t^n\, F(t)\, dt}{\int_0^\infty F(t)\, dt}. \tag{A.6}$$

If all these moments exist, then $F(s)$ can be written as a Taylor series in s These generate the positive integer moments of $F(t)$ by means of

$$\begin{aligned}
F(s) &= \int_0^\infty F(t)\, e^{-st}\, dt \\
&= \int_0^\infty F(t) \left(1 - st + \frac{s^2 t^2}{2!} - \frac{s^3 t^3}{3!} + \ldots\right) \\
&= \mathcal{F}(\infty) \left(1 - s\langle t \rangle + \frac{s^2}{2!}\langle t^2 \rangle - \frac{s^3}{3!}\langle t^3 \rangle + \ldots\right).
\end{aligned} \tag{A.7}$$

Thus the Laplace transform is a *moment generating function*, as it contains *all* the positive integer moments of the probability distribution $F(t)$.