Problem 9.10

- The number of particles in the initial state decays exponentially

as

$$N(t) = N_0 e^{-\frac{t}{\tau}}$$

- Half-life, $t_{1/2}$, is defined by

$$N(t_{1/2}) = \frac{1}{2} N_0$$

- Plugging in this condition and simplifying gives

$$N(t_{1/2}) = \frac{1}{2} N_0 = N_0 e^{-\frac{t_{1/2}}{\tau}}$$

$$\boxed{t_{1/2} = \log(2) \cdot \tau}$$

Problem 9.11

- The atom will decay to the ground state due to interactions with the electromagnetic fields. An expression for the lifetime is given in Griffiths by

$$\frac{1}{\tau} = \frac{\omega_0^2}{3\pi\epsilon_0 \hbar c^3} \left| \langle \psi_{final} | e\vec{r} | \psi_{initial} \rangle \right|^2 \qquad (1)$$

where $\omega_0 = \dfrac{E_{final} - E_{initial}}{\hbar}$

- We're interested in the rate of $n=2 \longrightarrow n=1$ so

$$\omega_0 = \frac{E_1 - E_2}{\hbar} = \frac{3E_1}{4\hbar} \quad ; \text{ where } E_1 = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = \frac{\hbar^2}{2ma^2}$$

- Next we need to calculate the inner product $\langle \psi_{final} | \vec{r} | \psi_{initial} \rangle$

recall: $\psi_{100} = \dfrac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$

$$\psi_{200} = R_{20} \cdot Y_0^0 = \left[\frac{1}{\sqrt{2a^3}}\left(1 - \frac{r}{2a}\right) e^{\frac{-r}{2a}}\right]\left[\frac{1}{\sqrt{4\pi}}\right]$$

$$\psi_{210} = R_{21} \cdot Y_1^0 = \left[\frac{1}{\sqrt{24a^3}}\left(\frac{r}{a}\right) e^{\frac{-r}{2a}}\right]\left[\sqrt{\frac{3}{4\pi}} \cos\theta\right]$$

$$\psi_{21\pm1} = R_{21}\, Y_1^{\pm1} = \left[\frac{1}{\sqrt{24a^3}}\left(\frac{r}{a}\right) e^{\frac{-r}{2a}}\right]\left[\mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}\right]$$

$\psi_{200} \longrightarrow \psi_{100}$

- using $\vec{r} = r\hat{r} = r\left[\cos\phi\sin\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k}\right]$

$$\langle \psi_{200} | \vec{r} | \psi_{100} \rangle = \int_0^\infty R_{20}(r)\, r\, R_{10}(r) \cdot r^2 dr \int \frac{1}{4\pi} \cdot \hat{r}\, d\Omega$$

The angular part integrates to zero by symmetry so

$$\langle \psi_{200} | \vec{r} | \psi_{100} \rangle = 0$$

- The three states $\psi_{21\pm1}$, ψ_{210} only differ by m_z but since we're considering the matrix element of $\vec{r}$ which has no prefered direction so the absolute value of each will be the same. Because we only care about the magnitude, we need only calculate once.

  - In Math:

$$\left|\langle \psi_{210} | \vec{r} | \psi_{100} \rangle\right| = \left|\langle \psi_{21\pm1} | \vec{r} | \psi_{100} \rangle\right| \qquad \text{by symmetry.}$$

$\psi_{210} \longrightarrow \psi_{100}$

$$\langle \psi_{210} | \vec{r} | \psi_{100} \rangle = \int_0^\infty \left[\frac{1}{\sqrt{24a^3}}\left(\frac{r}{a}\right) e^{\frac{-r}{2a}}\right] r \left[\frac{2}{\sqrt{a^3}} e^{\frac{-r}{a}}\right] r^2 dr \int_0^{2\pi}\int_0^{\pi} \left[\sqrt{\frac{3}{4\pi}} \cos\theta\right] \hat{r} \left[\frac{1}{\sqrt{4\pi}}\right] \sin\theta \, d\theta \, d\phi$$

- for the $\hat{x}$ and $\hat{y}$ components of the ϕ integral is zero, so we are only left with the $\hat{z}$ direction.
- I'll also scale the r integral by letting $r = \frac{2}{3} a \rho$

$$\langle \psi_{210} | \vec{r} | \psi_{100} \rangle = a \frac{2^4}{\pi 3^5 \sqrt{2}} \underbrace{\int_0^\infty \rho^4 e^{-\rho} d\rho}_{= 4!} \underbrace{\int_0^{2\pi} d\phi}_{= 2\pi} \underbrace{\int_0^1 (\cos\theta)^2 d(\cos\theta)}_{= \frac{1}{3}}$$

$$\langle \psi_{210} | \vec{r} | \psi_{100} \rangle = a \frac{2^8}{3^5 \sqrt{2}} = a \frac{\sqrt{2^{15}}}{3^5}$$

Combining all these relations and plugging into the formula for $\frac{1}{\tau}$ gives.

For $\underline{\psi_{200} \to \psi_{100}}$

$$\frac{1}{\tau} = 0 \quad \text{or} \quad \boxed{\tau \to \infty}$$ (ie. this transition will never occur)

for $\underline{\psi_{210} \to \psi_{100} \text{ and } \psi_{21\pm1} \to \psi_{100}}$

$$\frac{1}{\tau} = \frac{\left(\frac{3}{4\hbar} \cdot \frac{\hbar^2}{2ma^2}\right)^2}{3\pi \epsilon_0 \hbar c^3} \cdot \left[e^2 a^2 \frac{2^{15}}{3^{10}}\right]$$

after simplifying and plugging in values. This gives.

$$\boxed{\tau = 1.6 \times 10^{-9}\,s}$$

Problem 9.20

$$\vec{B} = B_{rf}\cos(\omega t)\,\hat{x} - B_{rf}\sin(\omega t)\,\hat{y} + B_0\,\hat{z}$$

a) The hamiltonian is given by $H = -\gamma \vec{B}\cdot\vec{S}$. Choosing the basis where $\begin{pmatrix}1\\0\end{pmatrix}$ is spin up in z direction

$\begin{pmatrix}0\\1\end{pmatrix}$ is spin down in z direction.

I can write the $\vec{S}$ operator with pauli matrices:

$$\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z) \quad \text{with} \quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

plugging in $\vec{S}$ and $\vec{B}$ gives:

$$H = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_{rf}(\cos\omega t + i\sin\omega t) \\ B_{rf}(\cos\omega t - i\sin\omega t) & -B_0 \end{pmatrix}$$

$$\boxed{H = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_{rf}e^{i\omega t} \\ B_{rf}e^{-i\omega t} & -B_0 \end{pmatrix}}$$

b) The time-dependent shrodenger equation is $i\hbar \frac{d\Psi}{dt} = H\Psi$

using $\Psi = \chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ gives

$$i\hbar \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{-\gamma\hbar}{2} \begin{pmatrix} B_0 & B_{rf} e^{i\omega t} \\ B_{rf} e^{-i\omega t} & -B_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \omega_0 a + \Omega e^{i\omega t} b \\ \Omega e^{-i\omega t} a - \omega_0 b \end{pmatrix}}$$

c) Solving the differential equation above directly could prove very difficult. But since we're given the answer we only need to check. To check for $a(t)$, say:

1. Differentiate the expression for $a(t)$
2. Equate the expression for $\dot{a}(t)$ you find to the expression from part (b).
3. Plug in values for $a(t)$ and $b(t)$ from the guess griffiths gives.
4. verify that all terms cancel and the equations are true.

d) The probability of being in down state is just

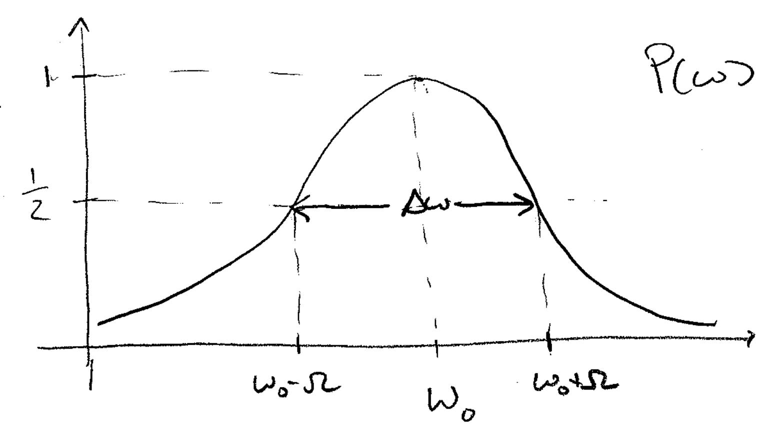
$$P(t) = |b(t)|^2$$

using $a_0 = 1$, $b_0 = 0$ This becomes:

$$P(t) = \left| \frac{i\Omega}{\omega'} \sin\left(\frac{\omega' t}{2}\right) \right|^2 \qquad \text{where } \omega' = \sqrt{(\omega-\omega_0)^2 + \Omega^2}$$

$$\boxed{P(t) = \left[\frac{\Omega^2}{(\omega-\omega_0)^2 + \Omega^2} \right] \sin^2\left(\frac{\omega' t}{2}\right)}$$

e)



$$P(\omega) = \frac{\Omega^2}{(\omega-\omega_0)^2 + \Omega^2}$$

- The height of peak is given by $P(\omega_0) = 1$
- The FWHM $= \Delta\omega$ satisfies: $P\left(\omega_0 \pm \frac{\Delta\omega}{2}\right) = \frac{1}{2} P(\omega_0)$

$$\frac{\Omega^2}{\left(\frac{\Delta\omega}{2}\right)^2 + \Omega^2} = \frac{1}{2} \implies$$

$$\boxed{\Delta\omega = 2\Omega}$$

f) Using $\gamma = g_{proton} \dfrac{e}{2M_{proton}}$ → with $g_{proton} = 5.59$

$B_0 = 10^4$ gauss

$B_{rf} = 10^{-2}$ gauss

We get the frequency (not angular frequency!) to be

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{\gamma B_0}{2\pi}$$

$$\boxed{\nu_0 = 4\times10^7 \text{ Hz}}$$

$$\Delta\nu = \frac{\Delta\omega}{2\pi} = \frac{2\Omega}{2\pi} = \frac{\gamma B_{rf}}{\pi}$$

$$\boxed{\Delta\nu = 85 \text{ Hz}}$$