PY 452 - Problem Set #8

#1 In class, the formula $P_{e\to m}(t) = \frac{|\langle m|V'(r)|e\rangle|^2}{\hbar^2}\left|\int_{-\infty}^{t} dt' \frac{e^{-t'^2/2\tau^2}}{\sqrt{2\pi\tau^2}} e^{i\omega_{me}t'}\right|$

was derived for this case. Let $\Gamma_{me} \equiv \langle m|V'(r)|e\rangle$.

$$\hookrightarrow P_{e\to m}(\infty) = \frac{|\Gamma_{me}|^2}{\hbar^2}\left|\int_{-\infty}^{\infty} \frac{dt'}{\sqrt{2\pi\tau^2}} e^{-t'^2/2\tau^2 + i\omega_{me}t'} dt'\right|^2$$

Complete the square for gaussian integral

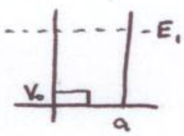
$$\hookrightarrow \boxed{P_{e\to m} = \frac{|\Gamma_{me}|^2}{\hbar^2} e^{-\omega_{me}^2\tau^2}}$$

- Less transition probability as τ increased, maximum probability of transition as $\tau \to \emptyset$.

- The perturbation results in this case are an expansion in $\frac{\Gamma_{me}}{\hbar}$, so for the perturbation to be small, we need $\left|\frac{\Gamma_{me}}{\hbar}\right| \ll 1$.

#2

$$V(x) = \begin{cases} V_0\,, & 0 \le x \le a/2 \\ 0\,, & a/2 \le x \le a \\ \infty\,, & \text{otherwise} \end{cases} \quad , \quad V_0 << E_1 \,,$$



$H' = V_0$

If we start in state ℓ, transition prob to m is

$$P_{\ell\to m}(t) = \frac{|\Gamma_{m\ell}|^2}{\hbar^2}\left|\int_0^t dt'\, e^{i\omega_{m\ell}t'}\right|^2 \quad , \quad E_i = \hbar\omega_i$$

$$\hookrightarrow P_{1\to 2}(t) = \frac{|V_{21}|^2}{\hbar^2}\left|\int_0^t dt'\, e^{i\omega_{21}t'}\right|^2 = \frac{4\,|V_{21}|^2}{(E_2-E_1)^2}\sin^2\left(\frac{(E_2-E_1)t}{2\hbar}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \Rightarrow E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2}$$

$$V_{21} = \frac{2V_0}{a}\int_0^{a/2} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)dx = \frac{4V_0}{3\pi}$$

$$\boxed{\hookrightarrow P_{1\to 2} = \frac{4\left(\frac{4V_0}{3\pi}\right)^2}{\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^2}\sin^2\left[\frac{3\pi^2\hbar t}{4ma^2}\right] = \left(\frac{16ma^2V_0}{9\pi^3\hbar^2}\right)^2\sin^2\left[\frac{3\pi^2\hbar t}{4ma^2}\right]}$$

#3

Again, if $H'(t) = f(t)V'$, then $P_{\ell\to m} = \frac{|V_{m\ell}|^2}{\hbar^2}\left|\int_{-\infty}^t dt'\, f(t')\, e^{i\omega_{m\ell}t'}\right|^2$

$H'(t) = \lambda(x-\frac{a}{2})\sin\omega t \Rightarrow f(t) = \sin(\omega t)$, $V' = \lambda(x - \frac{a}{2})$, let V turn on at $t=0$.

a)
$$P_{1\to 2} = \underbrace{\frac{|V_{21}|^2}{\hbar^2}}_{①}\underbrace{\left|\int_0^t \sin(\omega t')\, e^{i\omega_{21}t'}\, dt'\right|^2}_{②}$$

$$①:\ V_{21} = \frac{2\lambda}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\left(x-\frac{a}{2}\right)\sin\left(\frac{2\pi x}{a}\right)dx = -\frac{16a\lambda}{9\pi^2}$$

(2): $\int_0^t \sin(\omega t')\, e^{i\omega_{21}t'}\,dt' = \frac{1}{2i}\int_0^t \left(e^{i\omega t'} - e^{-i\omega t'}\right) e^{i\omega_{21}t'}\,dt'$

$$= \frac{-1}{2}\left[\frac{e^{i(\omega+\omega_{21})t}}{\omega+\omega_{21}} - \frac{e^{i(\omega_{21}-\omega)t}}{\omega_{21}-\omega}\right]_0^t$$

Since ω_{21} is positive, the second term dominates here.

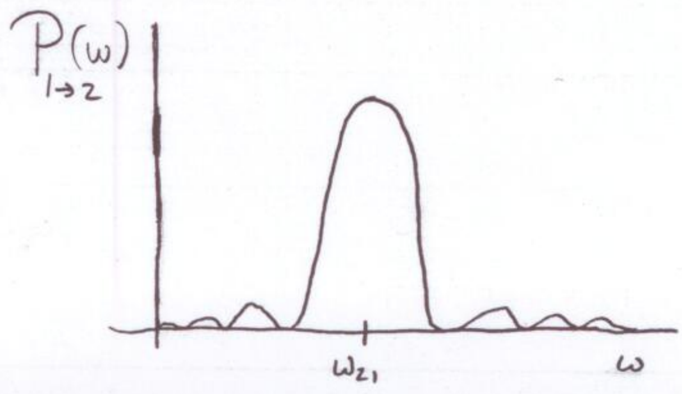
$$\hookrightarrow \boxed{P_{1\to 2} = \left(\frac{8a\lambda}{9\pi^2\hbar^2}\right)^2 \left(\frac{1}{\omega_{21}-\omega}\right)^2 \sin^2\left[\frac{(\omega_{21}-\omega)t}{2}\right]}$$

b) $V_{31} = \frac{2\lambda}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)(x - a/2)\sin\left(\frac{3\pi x}{a}\right)dx = 0$

$$\hookrightarrow \boxed{P_{1\to 3} = 0}$$

c) As $\omega \to 0$, nothing interesting happens, $P_{1\to 2} = \left(\frac{8a\lambda}{9\pi^2\hbar^2}\right)^2 \left(\frac{1}{\omega_{21}}\right)^2 \sin^2\left(\frac{\omega_{21}t}{2}\right)$

I believe the problem was intended to ask what happens when $\omega \to \omega_{21}$. In this case, you are driving the system at a frequency corresponding to an energy transition and thus this transition becomes likely to occur.



$\boxed{\#4}$ The particle in a harmonic oscillator for this problem was intended to have a charge q.

$$E = E_0 e^{-t^2/2\tau^2} \Rightarrow H' = -q\vec{E}\cdot\vec{r} = -qE_0 x e^{-t^2/2\tau^2}$$

$$\hookrightarrow P_{0\to1}(t) = \frac{q^2E_0^2|\langle 0|x|1\rangle|^2}{\hbar^2}\left|\int_{-\infty}^{\infty} e^{-t'^2/2\tau^2} e^{i\omega_{10}t'}dt'\right|^2$$

$$= \frac{q^2E_0^2|\langle 0|x|1\rangle|^2\, 2\pi\tau^2}{\hbar^2} e^{-\omega_{10}^2\tau^2}$$

We have calculated $\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{m}\,\delta_{n,m-1} + \sqrt{n}\,\delta_{n-1,m}\right)$ before, so $\langle 0|x|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}$

$$\hookrightarrow \boxed{P_{0\to1}(t) = \frac{q^2E_0^2\pi\tau^2}{\hbar m\omega} e^{-\omega_{10}^2\tau^2}}$$

a)

for $\omega_{10}\tau >> 1$, the transition becomes very unlikely, $P \to \emptyset$.

b)

for $\omega\tau \approx 1$, $P_{0\to1} = \frac{q^2E_0^2\pi\tau^2}{\hbar m\omega} e^{-1}$, again not sure of the significance of this.

No other transition can occur because $\langle 0|x|k\rangle = 0$ for all $k \geq 2$.