Problem Set 7

Py 452

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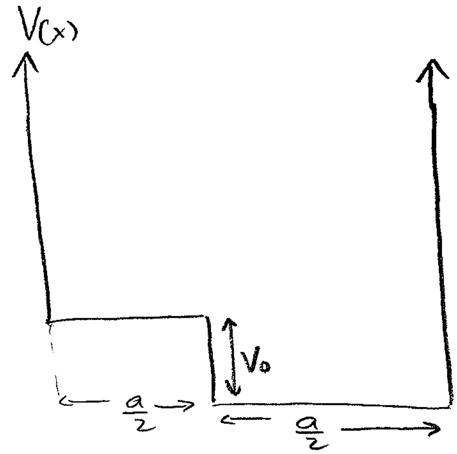
Problem 8.1



The book gives a nice argument in example 8.1 for the relation

$$\int_0^a p(x)\,dx = n\pi\hbar \qquad \text{where} \qquad p(x) = \sqrt{2m(E_n - V(x))}$$

- We need only plug in our form for $V(x)$, integrate, then solve for E_n.
- After plugging in $p(x)$ and $V(x)$ and separating the integral into 2 regions, we get

$$\sqrt{2m(E_n - V_0)}\left(\frac{a}{2}\right) + \sqrt{2m(E_n - 0)}\left(\frac{a}{2}\right) = n\pi\hbar$$

$$\sqrt{E_n - V_0} + \sqrt{E_n} = \sqrt{\frac{2\pi^2\hbar^2 n^2}{a^2 m^2}} = 2\sqrt{E_n^0} \quad , \text{where } E_0 = \frac{\pi^2\hbar^2 n^2}{2m^2a^2}$$

solving for E_n gives.

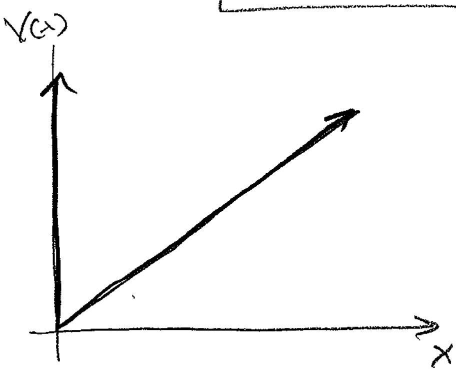
$$\boxed{E_n = E_n^0 + \frac{V_0}{2} + \frac{V_0^2}{16E_n^0}}$$

This differs from 1st order perterbation theory only by the 3rd term, $\frac{V_0^2}{E_n^0}$. This is negligible if V_0 is small or $E_n^0 \sim n^2$ is large.

Problem 8.5

a) Potential energy is

$$\boxed{V(x) = \begin{cases} mgx & x \geq 0 \\ 0 & x < 0 \end{cases}}$$



b) The time independant shrodenger equation in this region is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (mgx)\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(mgx - E)\psi$$

We convert this to the Airy equation form with the variable change,

$$z = \left(\frac{2m^2g}{\hbar^2}\right)^{1/3}\left(x - \frac{E}{mg}\right)$$

so,

$$\frac{d^2\psi}{dz^2} = z\psi$$

The $Bi(z)$ solution blows up at large z, so we're left with

$\psi = C \cdot Ai(z)$, where C is a constant that could be found through normalization. Replacing back in x, we get

$$\boxed{\psi(x) = C\, Ai\left[\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}\left(x - \frac{E}{mg}\right)\right]}$$

c) The $x \to \infty$ boundary condition was satisfied by dropping the $Bi(z)$ term. The left side boundary condition is,

$$\psi(0) = 0 \implies Ai\left[-\left(\frac{2}{\hbar^2 m g^2}\right)^{1/3} E\right] = 0$$

So the energy levels are defined by zero's of Airy function. The 1st four zero can be looked up easily. They are

$r_1 = -2.338$

$r_2 = -4.088$

$r_3 = -5.521$

$r_4 = -6.787$.

The n'th energy level will fit the relation,

$$r_n = -\left(\frac{2}{\hbar^2 m g^2}\right)^{1/3} E_n$$

Using $g = 9.8\ m/s^2$ and $m = .1\ kg$. This gives.

$$\boxed{\begin{aligned} E_1 &= 8.8 \times 10^{-23}\ J \\ E_2 &= 1.5 \times 10^{-22}\ J \\ E_3 &= 2.1 \times 10^{-22}\ J \\ E_4 &= 2.6 \times 10^{-22}\ J \end{aligned}}$$

d) plugging in electron mass gives

$$\boxed{E_1 = 1.1 \times 10^{-13}\ eV}$$

The Virial theorem says $2\langle T\rangle = \langle x \frac{dV}{dx} \rangle$

In our case, we have

$$2\langle T\rangle = \langle x (mg)\rangle = \langle mgx\rangle$$

Also

$$\langle H\rangle = \langle T\rangle + \langle mgx\rangle = E_1$$

putting together these equations and solving for $\langle x \rangle$ gives.

$$\langle x \rangle = \frac{2E_1}{3mg}$$

for an electron, $\boxed{\langle x \rangle = 1.4 \text{ mm}}$

Problem 8.6

In the book, example 8.3 gives the argument for

$$\int_0^{x_2} p(x)\,dx = (n-\tfrac{1}{4})\pi\hbar \quad \text{for} \quad p(x) = \sqrt{2m(E_n - V(x))}$$

and $V(x_2) = E_n$

Using $V(x) = mgx$ (thus $x_2 = \frac{E_n}{mg}$)

you can do the integral and solve for E_n to get,

$$\boxed{E_n = \left\{ \frac{9\pi^2 m g^2 \hbar^2}{8} (n-\tfrac{1}{4})^2 \right\}^{1/3}}$$

b) for $m = .1\,\text{kg}$, this gives.

$$\boxed{\begin{aligned} E_1 &= 8.7 \times 10^{-23}\ \text{J} \\ E_2 &= 1.5 \times 10^{-22}\ \text{J} \\ E_3 &= 2.1 \times 10^{-22}\ \text{J} \\ E_4 &= 2.6 \times 10^{-22}\ \text{J} \end{aligned}}$$

Very close to exact sol'n in problem 8.5!

c) Using the Viral Theorem as in part (d) of problem 8.5.

We have

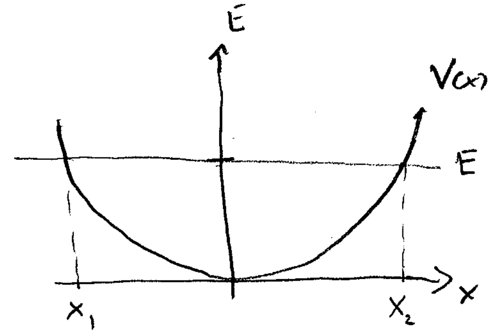
$$\langle x \rangle = \frac{2E_n}{3mg}$$

plugging in equation for E_n and setting $\langle x \rangle = 1m$ gives us n...

$$\boxed{n = 1.6 \times 10^{33}}$$ HUGE!

Problem 8.7

$$V(x) = \frac{1}{2}m\omega^2 x^2$$



Example 8.4 in book demonstrates the relation:

$$\int_{x_1}^{x_2} p(x)\,dx = \left(n - \frac{1}{2}\right)\pi\hbar \qquad \text{where } p(x) = \sqrt{2m(E_n - V(x))}$$

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

x_1, x_2 are the classical turning points where $E_n = V(x_1) = V(x_2)$

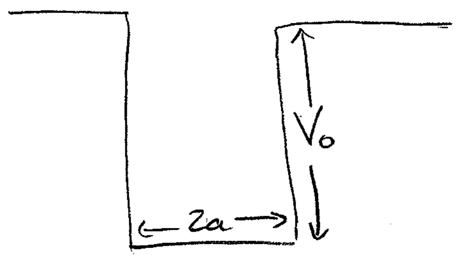
Performing the integral and solving for E_n gives

$$\boxed{E_n = \left(n - \frac{1}{2}\right)\hbar\omega}$$

This is the exact solution! (recall that we normally start counting the Harmonic oscillator states at $n=0$, not $n=1$)

Problem 8.16

a)



I just look up the infinity well energy to get

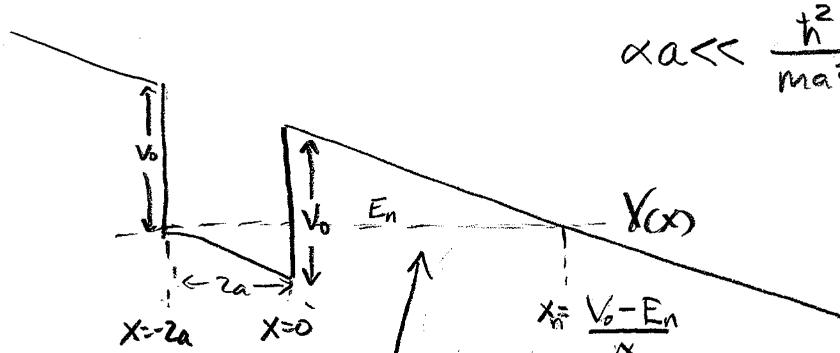
$$E_n = \frac{\pi^2 \hbar^2}{2m(2a)^2} n^2$$

so the ground state ($n=1$) is

$$\boxed{E_1 \approx \frac{\pi^2 \hbar^2}{8ma^2}} \quad \text{for } V_0 \gg \frac{\hbar^2}{ma^2}$$

b) with $H' = -\alpha x$ we get

$$\alpha a \ll \frac{\hbar^2}{ma^2} \ll V_0$$



particle can escape to the right

c) Using eq. 8.22 derived in Griffiths

$$\gamma = \frac{1}{\hbar}\int_0^{x_1} |p(x)|\,dx$$

where $p(x) = \sqrt{2m(E - V(x))}$

and the integration limits

$x = 0$

$x_1 = \dfrac{V_0 - E_1}{\alpha}$

are the left and right edge of the Barrier.

$$\gamma = \frac{\sqrt{2m}}{\hbar}\int_0^{x_1} \left|\sqrt{E_1 - V_0 + \alpha x}\right| dx$$

$$\gamma = \frac{\sqrt{2m\alpha}}{\hbar}\int_0^{x_1} \sqrt{x_1 - x}\,dx = \frac{\sqrt{2m\alpha}}{\hbar} \cdot -\frac{2}{3}(x_1 - x)^{\frac{3}{2}} \Big|_0^{x_1}$$

$$\gamma = \frac{\sqrt{8m\alpha}}{3\hbar}\, x_1^{3/2} = \frac{\sqrt{8m\alpha}}{3\hbar}\left(\frac{V_0 - E_1}{\alpha}\right)^{3/2}$$

if $V_0 \gg E_1$, then $V_0 - E_1 \approx V_0$

$$\boxed{\gamma \approx \frac{\sqrt{8mV_0^3}}{3\hbar\alpha}}$$

escape time is easy once we know γ. Semi-classically speaking the electron will hit the barrier every $\Delta t = \frac{4a}{v}$. Where v is its velocity, $\frac{1}{2}mv^2 = E_1$. The transmition probability is $e^{-2\gamma}$. So the average escape time is

$$\tau = \Delta t \, e^{-2\gamma}$$

substituting in values of v, E_1, and γ gives

$$\boxed{\tau = \frac{8ma^2}{\pi\hbar} \exp\left[\frac{2\sqrt{8mV_0^2}}{3\hbar\alpha}\right]}$$

d) We use the numbers Griffiths gives. The result is so huge that only the exponential factor really matters.

$$\tau = (10^{-19}) \exp\left[10^5\right] = 10^{-19} \cdot 10^{(\log e)\cdot 10^5} \approx 10^{-19} \cdot 10^{10^4} \, s$$

$$\boxed{\tau \approx 10^{10^4} \, s = 10^{10{,}000} \, s}$$

This is only an estimate, but the point is that it is huge!! The age of the universe is about 10^{17}s, thousands of orders smaller!!