PY 452 - Problem Set 4

#1 $H' = \lambda x^4$, $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle$

$\hookrightarrow E_n^1 = \lambda \langle n | x^4 | n \rangle$, $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$$\hookrightarrow E_n^1 = \frac{\lambda\hbar^2}{4m^2\omega^2}\left\langle n \middle| \begin{matrix} a_+a_+a_-a_- + a_+a_-a_+a_- + a_+a_-a_-a_+ \\ + a_-a_-a_+a_+ + a_-a_+a_-a_+ + a_-a_+a_+a_- \end{matrix} \middle| n \right\rangle$$

only equal numbers of a_+ and a_-

$$\hookrightarrow \boxed{E_n^1 = \frac{\lambda\hbar^2}{4m^2\omega^2}\left[n(n-1) + n^2 + n(n+1) + (n+1)(n+2) + (n+1)^2 + n(n+1)\right]}$$

$$E_n^2 = \sum_{m\neq n} \frac{|\langle m | \lambda x^4 | n \rangle|^2}{E_n^0 - E_m^0}$$

$$\hookrightarrow \langle m | x^4 | n \rangle = \frac{\hbar^2}{4m^2\omega^2}\langle m | a_+^4 + a_-^4 + a_+a_+a_+a_- + a_+a_+a_-a_+$$

$$+ a_+a_-a_+a_+ + a_-a_+a_+a_+ + a_-a_-a_-a_+$$

$$+ a_-a_-a_+a_- + a_-a_+a_-a_- + a_+a_-a_-a_- | n \rangle$$

$$= \frac{\hbar^2}{4m^2\omega^2}\Big[\sqrt{(n+1)(n+2)(n+3)(n+4)}\,\langle m | n+4 \rangle + \sqrt{n(n-1)(n-2)(n-3)}\,\langle m | n-4 \rangle$$

$$+ \langle m | n+3 \rangle \left(n\sqrt{(n+1)(n+2)} + [n+1]\sqrt{(n+1)(n+2)} + [n+2]\sqrt{(n+1)(n+2)} + [n+3]\sqrt{(n+1)(n+2)}\right)$$

$$+ \langle m | n-3 \rangle \left([n+1]\sqrt{n(n-1)} + n\sqrt{n(n-1)} + [n-1]\sqrt{n(n-1)} + [n-2]\sqrt{n(n-1)}\right)\Big]$$

$$= \frac{\hbar^2}{4m^2\omega^2}\Big[\sqrt{(n+1)(n+2)(n+3)(n+4)}\,\delta_{m,n+4} + \sqrt{n(n-1)(n-2)(n-3)}\,\delta_{m,n-4}$$

$$+ (4n+6)\sqrt{(n+1)(n+2)}\,\delta_{m,n+3} + (4n-2)\sqrt{n(n-1)}\,\delta_{m,n-3}\Big]$$

Note on #1

My solution is not correct for a few of the lowest energy states. For part a, the given solution is valid for $n \geq 2$ and for $n \geq 4$ for part b.

This is because for low n, you run into cases where a_- operates on $|0\rangle$ giving a zero contribution.

For example, the $a_+ a_+ a_- a_-$ term gives no energy shift contribution to the state $|1\rangle$ since

$$a_+ a_+ a_- a_- |1\rangle = a_+ a_+ a_- |0\rangle = \emptyset.$$

$$\hookrightarrow E_n^2 = \frac{\lambda^2 \hbar^2}{4m^2\omega^2} \sum_{m\neq n} \frac{\left[(n+1)(n+2)(n+3)(n+4)\delta_{m,n+4} + n(n-1)(n-2)(n-3)\delta_{m,n-4} + (4n+6)^2(n+1)(n+2)\delta_{m,n+3} + (4n-2)^2(n-1)n\delta_{m,n-3}\right]}{(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega}$$

$$= \frac{\lambda^2 \hbar}{4m^2\omega^3}\left[\frac{(n+1)(n+2)(n+3)(n+4)}{n-(n+4)} + \frac{n(n-1)(n-2)(n-3)}{n-(n-4)} + \frac{(4n+6)^2(n+1)(n+2)}{n-(n+3)} + \frac{(4n-2)^2 n(n-1)}{n-(n-3)}\right]$$

$$\boxed{= \frac{-5\lambda^2\hbar}{6m^2\omega^3}(2n+1)(7n[n+1]+9)}$$

6.6 $\psi_\pm^0 = \alpha_\pm \psi_a^0 + \beta_\pm \psi_b^0$

a) $$\langle \psi_+^0 | \psi_-^0 \rangle = \langle (\alpha_+ \psi_a^0 + \beta_+ \psi_b^0) | (\alpha_- \psi_a^0 + \beta_- \psi_b^0) \rangle$$

$$= \alpha_+^*\alpha_- + \beta_+^*\beta_-$$

6.22

$$= \alpha_+^*\alpha_-\left[1 + \frac{(E_+^1 - W_{aa}^*)(E_-^1 - W_{aa})}{W_{ab}^* W_{ab}}\right]$$

6.27

$$= \frac{\alpha_+^*\alpha_-}{|W_{ab}|^2}\left[|W_{ab}|^2 + (E_+^1 - W_{aa}^*)(E_-^1 - W_{aa})\right]$$

$$= \frac{\alpha_+^*\alpha_-}{|W_{ab}|^2}\left[|W_{ab}|^2 + W_{aa}W_{bb} - |W_{ab}|^2 - W_{aa}(W_{aa}+W_{bb}) + W_{aa}^2\right]$$

$$= \boxed{0}$$

b) $$\langle \psi_+^0 | H' | \psi_-^0 \rangle = \alpha_+^*\alpha_- W_{aa} + \alpha_+^*\beta_- W_{ab} + \beta_+^*\alpha_- W_{ba} + \beta_+^*\beta_- W_{bb}$$

$$= \alpha_+^*\alpha_-\left[W_{aa} + W_{ab}\frac{(E_-^1 - W_{aa})}{W_{ab}} + W_{ba}\frac{(E_+^1 - W_{aa})}{W_{ab}^*} + W_{bb}\underbrace{\frac{(E_+^1 - W_{aa})(E_-^1 - W_{aa})}{W_{ab}^* W_{ab}}}_{=-1 \text{ from above}}\right]$$

$$= \alpha_+^*\alpha_-\left[W_{aa} + E_-^1 - W_{aa} + E_+^1 - W_{aa} - W_{bb}\right] = \boxed{0}$$

c) $\langle \psi_\pm^0 | H' | \psi_\pm^0 \rangle = \alpha_\pm^* \alpha_\pm \langle a|H'|a\rangle + \alpha_\pm^* \beta_\pm \langle a|H'|b\rangle + \beta_\pm^* \alpha_\pm \langle b|H'|a\rangle + \beta_\pm^* \beta_\pm \langle b|H'|b\rangle$

$$= |\alpha_\pm|^2 \left[W_{aa} + W_{ab} \frac{(E_\pm^1 - W_{aa})}{W_{ab}} \right] + |\beta_\pm|^2 \left[W_{ba} \frac{(E_\pm^1 - W_{bb})}{W_{ba}} + W_{bb} \right]$$

$$= |\alpha_\pm|^2 E_\pm^1 + |\beta_\pm|^2 E_\pm^1 = \boxed{E_\pm^1}$$

6.7 a) $-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E\psi \Rightarrow \psi = Ae^{ikx} + Be^{-ikx}, \; k = \frac{\sqrt{2mE}}{\hbar}$

$\psi(x+L) = \psi(x) \Rightarrow kL = 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$

$\hookrightarrow \psi_n = Ae^{i\left(\frac{2\pi n}{L}\right)x}, \; \int_0^L |\psi_n|^2 dx = 1 \Rightarrow A = \frac{1}{\sqrt{L}}$

$\hookrightarrow E_n = \frac{\hbar^2 k^2}{2m} = \boxed{\frac{2}{m}\left(\frac{n\pi\hbar}{L}\right)^2}$ and $\boxed{\psi_n = \frac{1}{\sqrt{L}} e^{i\left(\frac{2\pi n}{L}\right)x}}$

$\boxed{n = 0, \pm 1, \pm 2, \ldots}$

b)

$$H' = -V_0 e^{-x^2/a^2} \quad (a \ll L)$$

$$\hookrightarrow W_{nn} = W_{-n,-n} = \frac{-V_0}{L} \int_{-\infty}^{\infty} e^{-x^2/a^2} dx = \frac{-V_0}{L} a\sqrt{\pi}$$

limits from hint

$$W_{n,-n} = \frac{-V_0}{L} \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{-i\frac{4\pi n x}{L}} dx = \frac{-V_0}{L} a\sqrt{\pi}\, e^{-\left(\frac{2\pi n a}{L}\right)^2}$$

complete square $\Rightarrow$ gaussian integra

$$\hookrightarrow E_\pm^1 = \frac{1}{2}\left[W_{nn} + W_{-n-n} \pm \sqrt{(W_{nn} - W_{-n-n})^2 + 4|W_{n,-n}|^2} \right] = \boxed{-\sqrt{\pi}\frac{V_0 a}{L}\left(1 \mp e^{-\left(\frac{2\pi n a}{L}\right)^2}\right)}$$

c) $\beta_{\pm} = \alpha \dfrac{(E_{\pm}^{1} - W_{nn})}{W_{n,-n}} = \alpha \dfrac{\pm\sqrt{\pi}\,/\left(\frac{V_0 a}{L}\right) e^{-\left(\frac{2\pi n a}{L}\right)^2}}{-\sqrt{\pi}\left(\frac{V_0 a}{L}\right) e^{-\left(\frac{2\pi n a}{L}\right)^2}} = \mp\alpha$

$\hookrightarrow \psi_+ = \alpha\psi_n + (-\alpha)\psi_{-n} = \alpha \frac{1}{\sqrt{L}}\left(e^{\frac{2\pi i n x}{L}} - e^{\frac{-2\pi i n x}{L}}\right) = i\alpha\frac{2}{\sqrt{L}}\sin\left(\frac{2\pi n x}{L}\right)$

$\hookrightarrow \boxed{\psi_+ = i\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi n x}{L}\right)}$ (normalization)

$\psi_- = \alpha\psi_n + \alpha\psi_{-n} = \boxed{\sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n x}{L}\right)}$

$\hookrightarrow E_+^1 = \langle\psi_+|H'|\psi_+\rangle = -\frac{2V_0}{L}\int_{-\infty}^{\infty} e^{-x^2/a^2}\sin^2\left(\frac{2\pi n x}{L}\right)dx$

$= \boxed{-\sqrt{\pi}\,\frac{V_0 a}{L}\left(1 - e^{-4a^2n^2\pi^2/L^2}\right)}$

$E_-^1 = \langle\psi_-|H'|\psi_-\rangle = -\frac{2V_0}{L}\int_{-\infty}^{\infty} e^{-x^2/a^2}\cos^2\left(\frac{2\pi n x}{L}\right)dx$

$= \boxed{-\sqrt{\pi}\,\frac{V_0 a}{L}\left(1 + e^{-4a^2n^2\pi^2/L^2}\right)}$

Which are indeed equal to the corrections calculated in part b.

d) We need operator A s.t. $[A, H^0] = [A, H'] = 0$
Consider the parity operator $A: x \mapsto -x$, We know $[A, H^0] = 0$ and clearly $[A, H'] = 0$ since H' only depends on x^2.
The eigenstates are the set of even and odd functions with respective distinct eigenvalues 1 and -1.
The states we constructed in part c are

precisely the even and odd combinations of ψ_n and ψ_{-n}. So the theorem works with A being the parity operator in this case.

$\boxed{\#4}$ inside the region $r<R$, we have $V_0+V' = \frac{-3e^2}{2R^3}\left(R^2-\frac{r^2}{3}\right)$

$\hookrightarrow V' = \frac{-3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r}$ for $r<R$ and $\emptyset$ otherwise

$\hookrightarrow E^1_{100} = \langle 1\,0\,0 | V' | 1\,0\,0 \rangle$

$$= \frac{4\pi}{\pi a_0^3}\int_0^R \left(\frac{-3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r}\right) e^{-2r/a_0}\, r^2\, dr$$

$$\boxed{= \frac{e^2}{2a_0R^3}\left[3a_0^3 - 3a_0R^2 + 2R^3 - 3a_0 e^{-2R/a_0}(a_0+R)^2\right]}$$

$\hookrightarrow E^1_{200} = \langle 2\,0\,0 | V' | 2\,0\,0 \rangle$

$$= \frac{4\pi}{32\pi a_0^3}\int_0^R \left(\frac{-3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r}\right)\left(2-\frac{r}{a_0}\right)^2 e^{-r/a_0}\, r^2\, dr$$

$$\boxed{= \frac{e^2 e^{-R/a_0}}{8a_0^2R^3}\left[2a_0 e^{R/a_0}(84a_0^3 - 6a_0R^2 + R^3) - 3(56a_0^4 + 56a_0^3R + 24a_0^2R^2 + 6a_0R^3 + R^4)\right]}$$

$\hookrightarrow E^1_{210} = \langle 2\,1\,0 | V | 2\,1\,0 \rangle$

$$= \frac{2\pi}{32\pi a_0^3}\int_0^\pi\int_0^R \left(\frac{-3e^2}{2R} + \frac{e^2r^2}{2R^3} + \frac{e^2}{r}\right)\left(\frac{r}{a_0}\right)^2 e^{-r/a_0} \cos^2\theta \sin\theta\, r^2\, dr\, d\theta$$

$$\boxed{= \frac{e^2}{8a_0^2R^3}\left[2a_0(60a_0^3 - 6a_0R^2 + R^3) - e^{-R/a_0}(120a_0^4 + 120a_0^3R + 48a_0^2R^2 + 10a_0R^3 + R^4)\right]}$$