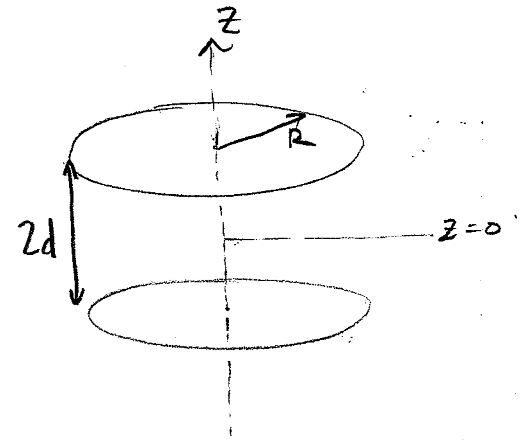
Problem Set 3

PY452

Written by Alan Gabel

Problem 1



- For a single coil at $z=0$ we have

$$\tilde{B}(z) = \frac{A}{[R^2+z^2]^{3/2}}$$

- In our case we have one coil displaced up by d and another down by d. So the magnetic field along the z axis is

$$B(z) = \tilde{B}(z-d) + \tilde{B}(z+d)$$

$$B(z) = \frac{A}{[R^2+(z-d)^2]^{3/2}} + \frac{A}{[R^2+(z+d)^2]^{3/2}}$$

$$B(z) = \frac{A}{R^3}\left\{ \frac{1}{[1+(z-\beta)^2]^{3/2}} + \frac{1}{[1+(z+\beta)^2]^{3/2}} \right\}$$

where $\beta = \frac{d}{R}$, the ratio we want to find.

- We want $B(z) = B(0) + az^4 + O(z^5)$. That is, the smallest power of z in a taylor series expansion is z^4.

- By symmetry, I know that $B(z)$ is even, so the z and z^3 terms will be zero. So we have

$$B(z) = B(0) + B'(0)\,z + \frac{1}{2!}B''(0)\,z^2 + \frac{1}{3!}B'''(0)\,z^3 + \frac{1}{4!}B''''(0)\,z^4 + O(z^5)$$

by symmetry:

$$B(z) = B(0) + \frac{1}{2!}B''(0)\,z^2 + \frac{1}{4!}B''''(0)\,z^4 + O(z^6)$$

- So we must choose $q = \frac{d}{R}$ so that $B''(0) = 0$.

After calculation:

$$B''(z=0) = \frac{-3R^3}{A}\left[\frac{2-8q^2}{[1+q^2]^{7/2}}\right] = 0$$

so $\boxed{q = \frac{d}{R} = \frac{1}{2}}$

Problem 2

The unperturbed wavefunction and energy for the n^{th} state is

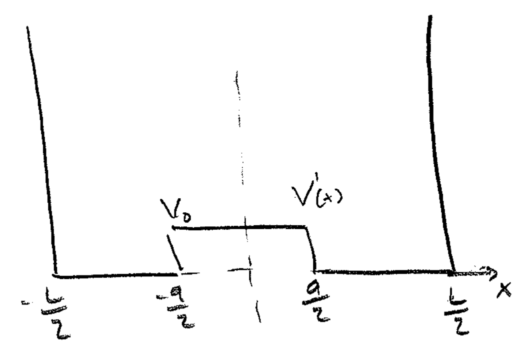
$$\psi_n^\circ(x) = \sqrt{\frac{2}{L}} \sin\left[\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right]$$

$$E_n^\circ = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

1^{st} Potential

$$H = H^\circ + V'(x)$$

$$V'(x) = \begin{cases} V_0 & -\frac{a}{2} < x < \frac{a}{2} \\ 0 & \text{else} \end{cases}$$



a) The first order energy corrections are given by

$$E_n' = \langle \psi_n^\circ | V' | \psi_n^\circ \rangle$$

$$E_n' = \int_{-\frac{L}{2}}^{\frac{L}{2}} V'(x) |\psi_n^\circ(x)|^2 dx$$

$$\boxed{E_n' = \frac{V_0}{L} \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin^2\left[\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right] dx = \frac{V_0 a}{L}\left[1 - \frac{L\cos(n\pi)\sin\left(\frac{an\pi}{L}\right)}{an\pi}\right]}$$

b) In general, the 1st order wavefunction correction is given by

$$|\psi_n^1\rangle = \sum_{m\neq n} \frac{\langle\psi_m^0|V'|\psi_n^0\rangle}{(E_n^0 - E_m^0)} |\psi_m^0\rangle$$

The matrix element of V' is given by the integral:

$$\langle\psi_m^0|V'|\psi_n^0\rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi_m^0(x)\,\psi_n^0(x)\,V'(x)\,dx$$

$$\langle\psi_m^0|V'|\psi_n^0\rangle = \frac{2V_0}{L}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left[\frac{m\pi}{L}\left(x+\frac{L}{2}\right)\right]\sin\left[\frac{n\pi}{L}\left(x+\frac{L}{2}\right)\right]dx$$

Doing the integration and plugging into the formula gives the general result.

c) We must demand that the corrections are small compared with the unperturbed energy values.

so consider

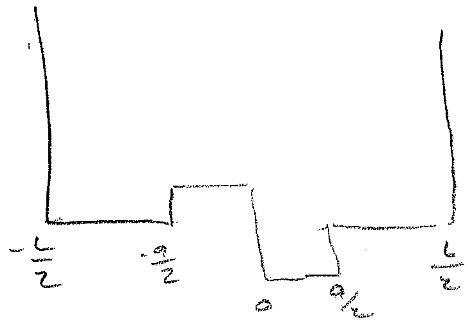
$$\frac{E_n^1}{E_n^0} = \frac{\frac{V_0 a}{L}\left[1 - \frac{L\cos(n\pi)\sin\left(\frac{an\pi}{L}\right)}{an\pi}\right]}{\frac{n^2\pi^2\hbar^2}{2mL^2}} < \frac{\frac{V_0 a}{L}}{\frac{n^2\pi^2\hbar^2}{2mL^2}}$$

$$= \frac{V_0 m a L}{\hbar^2} \cdot \frac{2}{\pi^2 n^2}$$

so our small dimensionless parameter is

$$\boxed{\frac{V_0 m a L}{\hbar^2} \ll 1}$$

d)



The procedure for the new perturbation is the same as for the old.

$$E_n^1 = \langle \psi_n^0 | V' | \psi_n^0 \rangle$$

$$E_n^1 = \int_{-\frac{L}{2}}^{\frac{L}{2}} V'(x) |\psi_n^0(x)|^2 \, dx$$

notice that $|\psi_n^0(x)|^2$ is an even function. $V'(x)$ is odd and the integral is over an even range, so it must be zero.

$$\boxed{E_n^1 = 0}$$

There is no 1st order correction of energy

The wavefunction correction is

$$|\psi_n^1\rangle = \sum_{m \neq n} \frac{\langle \psi_m^0 | V' | \psi_n \rangle}{(E_n - E_m)} |\psi_m^0\rangle$$

where

$$\langle \psi_m^0 | V' | \psi_n \rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi_m^0(x)\, \psi_n^0(x)\, V'(x)\, dx$$

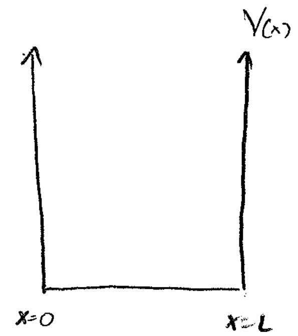
$$= \frac{2V_0}{L} \int_{-\frac{a}{2}}^{0} \sin\left(\frac{m\pi}{L}\left(x + \frac{L}{2}\right)\right) \sin\left(\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right) dx$$

$$- \frac{2V_0}{L} \int_{0}^{\frac{a}{2}} \sin\left(\frac{m\pi}{L}\left(x + \frac{L}{2}\right)\right) \sin\left(\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right) dx$$

By symmetry arguments we can say that $\psi_m^0(x)\psi_n^0(x)$ must be an odd function or the integral will be zero. So the only surviving terms are those where m is odd, n is even, or vice-versa.

. Plugging the integral into the sum gives the final answer.

Problem 6.3



$$V(x_1,x_2) = -L V_0\, \delta(x_1 - x_2)$$

The single particle wave functions are $\hat{\Psi}_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

a) These are boson's so the ground state is

$$\psi_1^0(x_1, x_2) = \psi_1(x_1)\psi_1(x_2)$$

$$\boxed{\psi_1^0(x_1, x_2) = \frac{2}{L} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right)}$$

The energy is

$$E_1^0 = \frac{\pi^2 \hbar^2}{2ma^2} + \frac{\pi^2 \hbar^2}{2ma^2}$$

$$\boxed{E_1^0 = \frac{\pi^2 \hbar^2}{ma^2}}$$

The first excited state is

$$\psi_2^\circ(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right]$$

$$\boxed{\psi_2^\circ(x_1, x_2) = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]}$$

Energy is

$$E_2^\circ = \frac{\pi^2\hbar^2}{2ma^2} + \frac{2^2\pi^2\hbar^2}{2ma^2}$$

$$\boxed{E_2^\circ = \frac{5\pi^2\hbar^2}{2ma^2}}$$

b) The first order energy correction is the expectation value of the perturbation, so

$$E_n^1 = \langle \psi_n^0 | V | \psi_n^0 \rangle$$

$$= \iint dx_1\, dx_2\, |\psi_n^0(x_1, x_2)|^2 \cdot -L V_0\, \delta(x_1 - x_2)$$

$$= -L V_0 \int_0^L dx_1\, |\psi_n^0(x_1, x_1)|^2$$

After doing the integral you get

ground state correction: $\boxed{E_1^1 = -\frac{3}{2} V_0}$

1st excited state correction: $\boxed{E_2^1 = -2V_0}$

10

Problem 6.5

a) The first order energy correction is

$$E_n^1 = \langle n | H' | n \rangle = -qE \langle n | x | n \rangle$$

rewriting x in terms of raising and lowering operators gives.

$$E_n^1 = -qE \cdot \sqrt{\frac{\hbar}{2m\omega}} \langle n | a_+ + a_- | n \rangle$$

the raising/lowering operators have no diagonal elements so

$$\boxed{E_n^1 = 0} \checkmark$$

The second order correction is

$$E_n^2 = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_m^0 - E_m^0}$$

$$= \frac{q^2 E^2}{\hbar \omega} \sum_{m \neq n} \frac{|\langle m | x | n \rangle|^2}{n - m}$$

We must calculate the matrix element $\langle m | x | n \rangle$. I'll again use the raising and lowering operators.

$$\langle m | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle m | a_+ + a_- | n \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n+1} \langle m | n+1 \rangle + \sqrt{n} \langle m | n-1 \rangle \right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n+1} \, \delta_{m,n+1} + \sqrt{n} \, \delta_{m,n-1} \right]$$

Plugging back in, we see that only 2 terms survive in the sum.

$$E_n^2 = \frac{q^2 \mathcal{E}^2}{\hbar\omega} \cdot \frac{\hbar}{2m\omega} \left\{ \frac{n+1}{n-(n+1)} + \frac{n}{n-(n-1)} \right\}$$

$$\boxed{E_n^2 = -\frac{q^2 \mathcal{E}^2}{2m\omega^2}}$$

b) The full hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - q\mathcal{E}x$$

We can "complete the square" to get

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left[x - \frac{q\mathcal{E}}{m\omega^2} \right]^2 - \frac{q^2 \mathcal{E}^2}{2m\omega^2}$$

Make the substitution $r = x - \frac{q\mathcal{E}}{m\omega^2}$ and notice that the same comutation relation, $[r, p] = i\hbar$, applies.

The hamiltonian is that of a harmonic oscillator shifted by a constant. The energy is

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{1}{2}\frac{q^2 E^2}{m\omega^2}$$

our perturbative correction was exact!