PY 452 Problem Set 2 Solutions

5.6 infinite square well, $\psi_\ell, \psi_n, (n \neq \ell)$ $\qquad \langle (x_1-x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 \rangle \langle x_2 \rangle$

a) distinguishable particles

$$\langle x_1^2 \rangle = \frac{2}{a}\int_0^a x_1^2 \sin^2\left(\frac{n\pi x_1}{a}\right) dx_1 \int_0^a |\psi_\ell(x_2)|^2 dx_2, \quad y = \frac{n\pi x_1}{a}$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} y^2 \sin^2(y)\, dy$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \cdot \frac{1}{24}\left(4n^3\pi^3 + 3\sin(2\pi n) - 6n\pi\left(\cos(2\pi n) + n\pi \sin(2\pi n)\right)\right)$$

$$= \boxed{a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)}$$

$\hookrightarrow \langle x_2^2 \rangle = \boxed{a^2\left(\frac{1}{3} - \frac{1}{2\ell^2\pi^2}\right)}$

$$\langle x_1 \rangle = \frac{2}{a}\int_0^a x_1 \sin^2\left(\frac{n\pi x_1}{a}\right) dx_1, \quad y = \frac{n\pi x_1}{a}$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y \sin^2(y)\, dy$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \cdot \frac{1}{8}\left(1 - \cos(2\pi n) + 2n^2\pi^2 - 2\pi n \sin(2\pi n)\right)$$

$$= \boxed{\frac{a}{2}}$$

$$\hookrightarrow \langle (x_1-x_2)^2 \rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2\ell^2\pi^2}\right) - 2\left(\frac{a}{2}\right)^2$$

$$= \boxed{a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{\ell^2}\right)\right]}$$

b+c) identical bosons and fermions

$$\langle (x_1-x_2)^2 \rangle_{\substack{\text{Boson}\\ \text{Fermion}}} = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 \rangle \langle x_2 \rangle \mp 2|\langle x \rangle_{n\ell}|^2 \quad \text{(eq. 5.21)}$$

$$\langle x \rangle_{n\ell} = \frac{2}{a}\int_0^a x \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{\ell\pi x}{a}\right) dx = \frac{1}{a}\int_0^a x\left[\cos\left(\frac{(n-\ell)\pi x}{a}\right) - \cos\left(\frac{(n+\ell)\pi x}{a}\right)\right] dx$$

$$= \frac{1}{a}\left[\left(\frac{a}{(n-\ell)\pi}\right)^2 \left(\cos((n-\ell)\pi)-1\right) - \left(\frac{a}{(n+\ell)\pi}\right)^2 \left(\cos((n+\ell)\pi)-1\right)\right]$$

$$= \frac{-8an\ell}{\pi^2(n^2-\ell^2)^2} \quad \text{if} \quad n+\ell = \text{odd}, \quad \phi \text{ if } n+\ell=\text{even}$$

↳
$$\boxed{\langle (x_1-x_2)^2 \rangle_{\substack{\text{Boson}\\ \text{Fermion}}} = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2}+\frac{1}{\ell^2}\right)\right] \mp \frac{128\,a^2n^2\ell^2}{\pi^4(n^2-\ell^2)^4}}$$

where the last term only appears if $n+\ell = \text{odd}$

5.23 $E = \frac{9}{2}\hbar\omega = (n_1+n_2+n_3+3/2)\,\hbar\omega \Rightarrow n_1+n_2+n_3 = 3$

i)

n_1	n_2	n_3	state
0	0	3	
0	3	0	$\langle 2,0,0,1,0,\ldots\rangle$ (3 distinct)
3	0	0	
0	1	2	
0	2	1	
1	2	0	$\langle 1,1,1,0,\ldots\rangle$ (6 distinct) ← most probable configuration
2	1	0	
1	0	2	
2	0	1	
1	1	1	$\langle 0,3,0,0,\ldots\rangle$ (1 distinct)

Possible energies are

E	$\frac{\hbar\omega}{2}$	$\frac{3\hbar\omega}{2}$	$\frac{5\hbar\omega}{2}$	$\frac{7\hbar\omega}{2}$
Prob	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

↳ Most Probable energy is $\frac{\hbar\omega}{2}$

b) for identical fermions, only choice is $n_1 = 0, n_2 = 1, n_3 = 2$

↳ State is $\langle 1,1,1,0,\ldots \rangle$, the only and most probable configuration.

Possible Single particle energies

E	$\frac{\hbar\omega}{2}$	$\frac{3\hbar\omega}{2}$	$\frac{5\hbar\omega}{2}$
Prob	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

↳ All equally likely.

c) for identical bosons, all 3 distinguishable configurations possible, but each is only one distinct state since bosons are indistinguishable.

$\langle 2,0,0,1,0,\ldots \rangle$, $\langle 1,1,1,0,\ldots \rangle$, $\langle 0,3,0,\ldots \rangle$ (All equally likely)

↳

E	$\frac{\hbar\omega}{2}$	$\frac{3\hbar\omega}{2}$	$\frac{5\hbar\omega}{2}$	$\frac{7\hbar\omega}{2}$
Prob	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

=> Most probable energy is $\frac{3\hbar\omega}{2}$

Lagrange Multipliers

a) $x + y = 100$, $f(x,y) = xy$, $g(x,y) - c = x + y - 100$

↳ $\Lambda = xy + \lambda \cdot (x + y - 100)$

$$\frac{\partial \Lambda}{\partial x} = y + \lambda = 0$$
$$\frac{\partial \Lambda}{\partial y} = x + \lambda = 0$$

$\Rightarrow \boxed{y = x} \Rightarrow \boxed{y = x = 50}$

↳ $\boxed{f(x,y) = 50^2 = 2500}$

b) $f(x,y) = x^2 + 4xy$, $g(x,y) - c = x^2 y - V = 0$

↳ $\Lambda = x^2 + 4xy + \lambda (x^2 y - V)$

$$\frac{\partial \Lambda}{\partial x} = 2x + 4y + 2\lambda xy = 0$$
$$\frac{\partial \Lambda}{\partial y} = 4x + \lambda x^2 = 0$$

$\Rightarrow x = 2y \Rightarrow \boxed{x = (2V)^{1/3}, \quad y = (V/4)^{1/3}}$

c) $f(x,y,z) = xyz$, $g_1(x,y,z) - c_1 = x^2+y^2+z^2-1=0$, $g_2(x,y,z) - c_2 = xy+yz+zx-1=0$

↳ $\Lambda = xyz + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(xy+yz+zx-1)$

↳ $\frac{\partial \Lambda}{\partial x} = yz + 2\lambda_1 x + \lambda_2(y+z) = 0$

$\frac{\partial \Lambda}{\partial y} = xz + 2\lambda_1 y + \lambda_2(x+z) = 0$

$\frac{\partial \Lambda}{\partial z} = xy + 2\lambda_1 z + \lambda_2(y+x) = 0$

Not sure how to reduce this to conditions on x, y, and z, if you figured it out, extra credit for you!

$\boxed{5.27}$

a) Percent error is $\frac{\ln(z!) - z\ln(z) + z}{\ln(z!)} \cdot 100$

↳ $\frac{\ln(10!) - 10\ln(10) + 10}{\ln(10!)} \cdot 100 = \boxed{13.76\%}$

b) I used Mathematica for this,

error for $z=89$ is 1.009%

error for $z=90$ is $.996\%$

So $\boxed{z=90}$ is the smallest integer with less than 1% error.

5.36

a) $dE = \frac{\hbar^2 k^2}{2m} \frac{V}{\pi^2} k^2 dk \longrightarrow dE = (\hbar c k) \frac{V}{\pi^2} k^2 dk$

$\hookrightarrow E_{tot} = \frac{\hbar c V}{\pi^2} \int_0^{k_f} k^3 dk = \frac{\hbar c V}{4\pi^2} k_f^4 = \boxed{\frac{\hbar c}{4\pi^2} (3\pi^2 N q)^{4/3} V^{-1/3}}$

b) $V = \frac{4}{3}\pi R^3 \Rightarrow E_{deg} = \frac{\hbar c}{3\pi R} \left(\frac{9}{4}\pi N q\right)^{4/3}$

$E_g = \frac{-3GN^2M^2}{R} \Rightarrow E_{tot} = \frac{\hbar c}{3\pi R}\left(\frac{9}{4}\pi N q\right)^{4/3} - \frac{3GN^2M^2}{R}$

Now to minimize energy $\frac{dE_{tot}}{dR} = 0 \Rightarrow \frac{\hbar c}{3\pi}\left(\frac{9}{4}\pi N q\right)^{4/3} = 3GN^2M^2$

at the critical point where $E_{tot} = 0$. Now we solve for N.

$\hookrightarrow N_c = \frac{15}{16}\sqrt{5\pi}\left(\frac{\hbar c}{G}\right)^{3/2} \frac{q^2}{M^3} = \boxed{2.04 \cdot 10^{57}}$

Mass of Sun $= 1.99 \cdot 10^{30}$ kg $\Rightarrow N_{sun} = \frac{1.99 \cdot 10^{30}\text{ kg}}{1.67 \cdot 10^{-27}\text{ kg}} = 1.19 \cdot 10^{57}$

$\frac{2.04}{1.19} = 1.71 \Rightarrow \boxed{M_{ch} \cong 1.71\, M_{sun}}$