Problem Set 11

PY 452

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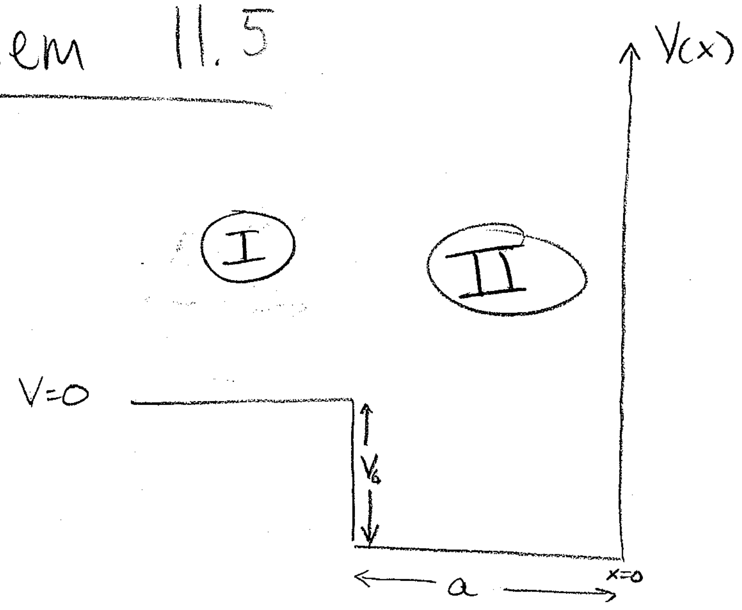
Problem 11.5



a)

- The wave function in each region is

$$\psi_I(x) = \underbrace{Ae^{ikx}}_{\text{incoming wave}} + \underbrace{Be^{-ikx}}_{\text{outgoing wave.}}$$

where $k = \frac{\sqrt{2mE}}{\hbar}$

$$\psi_{II}(x) = C\sin(k'x) + D\cos(k'x)$$

$k' = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

- Next we satisfy the boundary conditions at $x=0$ and $x=-a$.

<u>$x=0$</u>

$\psi(0)=0$ so ψ_{II} cannot have the cosine term.

Set $D=0$

continuity at $x=-a$

$$\psi(-a) = \psi_I(-a) = \psi_{II}(-a)$$

$$Ae^{-ika} + Be^{ika} = -C\sin(k'a) \qquad (i)$$

continous derivative at $x=-a$

$$\psi'(-a) = \psi_I'(-a) = \psi_{II}'(-a)$$

$$ikAe^{-ika} - ikBe^{ika} = k'C\cos(k'a) \qquad (ii)$$

- Solving these two equations for B gives

$$\boxed{B = Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right]}$$

✓

b) The amplitude of reflected wave is $|B|^2$

$$|B|^2 = BB^* = (AA^*)\left(\cancel{e^{-2ika}}\ \cancel{e^{2ika}}\right)\left[\frac{k-\cancel{ik'\cot(k'a)}}{k+\cancel{ik'\cot(k'a)}}\right]\left[\frac{k+\cancel{ik'\cot(k'a)}}{k-\cancel{ik\cot(k'a)}}\right]$$

$$|B|^2 = |A|^2$$

The amplitudes are the same.

c) The phase shift δ is defined as

$$e^{2i\delta} = -B$$

$$e^{2i\delta} = -e^{2ika}\left[\frac{\frac{k}{k'} - i\cot(k'a)}{\frac{k}{k'} + i\cot(k'a)}\right]$$

where I divided the numerator and denominator by k'

For a deep well $\frac{E}{V_0} \ll 1$ so $\frac{k}{k'} = \frac{\frac{\sqrt{2mE}}{\hbar}}{\frac{\sqrt{2m(E+V_0)}}{\hbar}} \ll 1$

we can drop the $\frac{k}{k'}$ terms to get

$$e^{2i\delta} = e^{-2ika} \Longrightarrow \boxed{\delta = -ka}$$

Problem 11.6

- From example (11.3), the partial wave amplitudes are

$$a_\ell = i\,\frac{j_\ell(ka)}{k\,h_\ell^{(1)}(ka)} = \frac{i}{k}\,\frac{j_\ell(ka)}{j_\ell(ka) + i\,n_\ell(ka)}$$

where j_ℓ and n_ℓ are spherical Bessel functions.

- The connection between a_ℓ and δ is

$$a_\ell = \frac{e^{i\delta_\ell}}{k}\sin\delta_\ell$$

$$\frac{i}{\cancel{k}}\,\frac{j_\ell(ka)}{j_\ell(ka) + i\,n_\ell(ka)} = \frac{e^{i\delta_\ell}}{\cancel{k}}\sin\delta_\ell$$

- Now we only need to solve this equation for δ_ℓ. It seems pretty nasty but gets simpler if we look at the real and imaginary parts separately.

$$\frac{j_\ell n_\ell}{j_\ell^2 + n_\ell^2} + i\,\frac{j_\ell^2}{j_\ell^2 + n_\ell^2} = \sin\delta_\ell\cos\delta_\ell + i\sin^2\delta_\ell$$

by inspection,

$$\boxed{\begin{aligned} \sin \delta_\ell &= \frac{j_\ell}{\sqrt{j_\ell^2 + n_\ell^2}} \\ \cos \delta_\ell &= \frac{n_\ell}{\sqrt{j_\ell^2 + n_\ell^2}} \\ &\text{or} \\ \tan \delta_\ell &= \frac{j_\ell(ka)}{n_\ell(ka)} \end{aligned}}$$

All these answers are equivalent

- To look at high and low energy limits, I first will rewrite the spherical bessel functions, j_ℓ and n_ℓ, in terms of the ordinary bessel functions J_α and Y_α.

- I looked up,

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$$

$$n_\ell(x) = \sqrt{\frac{\pi}{2x}}\, Y_{\ell+\frac{1}{2}}(x)$$

now or equation for δ_ℓ is

$$\tan \delta_\ell = \frac{J_{\ell+\frac{1}{2}}(ka)}{Y_{\ell+\frac{1}{2}}(ka)}$$

• The asymptotics of bessel functions are well known. I looked them up:

for $x \ll \sqrt{n+1}$: $J_n(x) \rightarrow \frac{1}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n$

$$Y_n(x) \rightarrow -\frac{\Gamma(n)}{\pi} \left(\frac{2}{x}\right)^n$$

For $x \gg \left|n^2 - \frac{1}{4}\right|$: $J_n(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$Y_n(x) \rightarrow -\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} + \frac{\pi}{4}\right)$$

so if we define the low energy limit as

$$ka \ll \sqrt{(\ell+\tfrac{1}{2})+1}$$

$$\boxed{ka \ll \sqrt{\ell + \tfrac{3}{2}}}$$

Then

$$\tan \delta_\ell \approx \frac{-\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})} \left(\frac{ka}{2}\right)^{2(\ell+\frac{1}{2})}$$

using $\tan \delta_\ell \approx \delta_\ell$ we get

$$\boxed{\delta_\ell \approx \frac{-\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})} \left(\frac{ka}{2}\right)^{2\ell+1}}$$

Define the high energy limit

$$ka >> \left| (\ell + \tfrac{1}{2})^2 - \tfrac{1}{4} \right| = \left| \ell^2 + \ell \right|$$

$$\boxed{ka >> \ell(\ell+1)}$$

Then

$$\tan \delta_\ell \approx \frac{\sqrt{\frac{2}{\pi(ka)}} \sin\left(ka - \frac{(\ell+\frac{1}{2})\pi}{2} + \frac{\pi}{4}\right)}{-\sqrt{\frac{2}{\pi(ka)}} \cos\left(ka - \frac{(\ell+\frac{1}{2})\pi}{2} + \frac{\pi}{4}\right)}$$

$$\tan \delta_\ell \approx \tan\left[-\left(ka - \frac{\ell\pi}{2}\right)\right]$$

$$\boxed{\delta_\ell \approx \frac{\ell\pi}{2} - ka}$$

Problem 11.12

- The scattering amplitude $f(\theta)$ is found by Griffiths in example 11.5. The result is

$$f(\theta) = \frac{-2m\beta}{\hbar^2(\mu^2 + \kappa^2)} \quad \text{where } \kappa = 2k\sin\left(\frac{\theta}{2}\right)$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

The potential is $V(r) = \beta \dfrac{e^{-\mu r}}{r}$

- I eventually want my expression to contain E so I'll substitute in now.

$$f(\theta) = \frac{-2m\beta}{\hbar^2\left(\mu^2 + 4\left(\frac{2mE}{\hbar^2}\right)\sin^2\left(\frac{\theta}{2}\right)\right)} = \frac{-2m\beta}{\hbar^2\mu^2}\left[\frac{1}{1 + \frac{8mE}{\hbar^2\mu^2}\sin^2\left(\frac{\theta}{2}\right)}\right]$$

The total cross section is given by

$$\sigma = \int |f(\theta)|^2 d\Omega = 2\pi \cdot \left(\frac{-2m\beta}{\hbar^2\mu^2}\right)^2 \int_0^{\pi} \frac{\sin\theta \, d\theta}{\left(1 + \frac{8mE}{\hbar^2\mu^2}\sin^2\frac{\theta}{2}\right)^2}$$

- I want to make this a dimensionless integral.

let $y = \sqrt{\dfrac{8mE}{\hbar^2\mu^2}} \sin\dfrac{\theta}{2}$

$$dy = \frac{1}{2}\sqrt{\frac{8mE}{\hbar^2\mu^2}} \cos\left(\frac{\theta}{2}\right) d\theta$$

also using $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ The equation becomes.

$$\sigma = \frac{2\pi m \beta^2}{\hbar^2 \mu^2 E} \int_0^{\sqrt{\frac{8mE}{\hbar^2\mu^2}}} \frac{2y\,dy}{(1+y^2)^2}$$

The integral is now easy.

$$\sigma = \frac{2\pi m\beta^2}{\hbar^2\mu^2 E}\left[\,1 + \frac{1}{1+\frac{8mE}{\hbar^2\mu^2}}\right]$$

$$\sigma = \frac{2\pi m\beta^2}{\hbar^2\mu^2 E} \cdot \frac{\frac{8mE}{\hbar^2\mu^2}}{1+\frac{8mE}{\hbar^2\mu^2}}$$

$$\boxed{\sigma = \frac{16\pi m^2\beta^2}{\hbar^4\mu^4} \cdot \frac{1}{1+\frac{8mE}{\hbar^2\mu^2}}}$$

Problem 11.15

Soft sphere: $V(r) = \begin{cases} V_0 & r \leq a \\ 0 & r > a \end{cases}$

In the 2nd order born approximation,

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\, \psi_0(\vec{r})\, d^3\vec{r}_0$$

$$+ \left(\frac{m}{2\pi\hbar^2}\right)^2 \iint \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \cdot \frac{e^{ik|\vec{r}_0-\vec{r}_1|}}{|\vec{r}_0-\vec{r}_1|} \cdot V(\vec{r}_0) V(\vec{r}_1)\, \psi_0(\vec{r}_1)\, d^3\vec{r}_0\, d^3\vec{r}_1$$

I'll start with some simplifications.

- let $\psi_0(\vec{r}) = e^{ikz}$
- since $|\vec{r}| \gg |\vec{r}_0|$, $\dfrac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \approx \dfrac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{r}_0}$, $\vec{k} = k\hat{r}$
- because this is low energy

$$k|\vec{r}-\vec{r}_0| \approx 0$$

$$k|\vec{r}_0-\vec{r}_1| \approx 0$$

I drop the exponentials

• substitute in potential and only integrate over sphere $r<a$

$$\psi(\vec{r}) = e^{ikz} + \frac{e^{ikr}}{r}\left\{ \frac{-mV_0}{2\pi\hbar^2}\int_{r_0<a} d^3\vec{r}_0 + \left(\frac{mV_0}{2\pi\hbar^2}\right)^2 \int_{r_0<a}\int_{r_1<a} \frac{d^3\vec{r}_0\, d^3\vec{r}_1}{|\vec{r}_1-\vec{r}_0|}\right\}$$

The term in brackets is the scattering amplitude.

• Chose axis so $|\vec{r}_1-\vec{r}_0| = r_1^2 + r_0^2 - 2rr_0\cos\theta_0$

$$f(\theta) = \frac{-mV_0}{2\pi\hbar^2}\,\frac{4}{3}\pi a^3 + \left(\frac{mV_0}{2\pi\hbar^2}\right)^2 \int_{r_1<a} d^3\vec{r}_1 \int_{r_0<a} \frac{r_0^2\, dr_0\, d(\cos\theta_0)\, d\phi}{\sqrt{r_0^2+r_1^2-2rr_0\cos\theta_0}}$$

do the ϕ_0 the θ_0 integrals then do r_0. The final result is

$$\boxed{f(\theta) = -\frac{2mV_0a^3}{3\hbar^2} + \frac{8m^2V_0^2a^5}{15\hbar^4}}$$