PY 452 Midterm

Date: Thursday October 14, 2010, 9:30am–11:00am

A SINGLE 11" x 8.5" sheet of paper with your notes is ALLOWED. Additional notes, calculators, or any electronic devices are prohibited. PLEASE EXPLAIN YOUR WORK CLEARLY to maximize the possibility of partial credit.

1. Consider a gas of N free electrons that is confined to a two-dimensional $L \times L$ square box of area $A = L^2$. The potential is zero inside the box and infinite outside.

   (a) Calculate the Fermi momentum of the gas in terms of the areal density $\sigma = N/A$ and other fundamental constants.

   (b) From the Fermi momentum, determine the Fermi energy of the gas and estimate its value numerically under the assumption that there is one electron per square Angstrom.

   (c) How does the pressure of the electron gas depend on σ? Explain your result physically.

   For half credit work this problem for N electrons in a three-dimensional $L \times L \times L$ cube.

2. Consider the one-dimensional simple harmonic oscillator that experiences a quartic perturbation. The Hamiltonian for the system is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \equiv H^0 + \lambda x^4$.

   (a) Calculate the correction to the ground-state energy to first-order in perturbation theory. (For this section, you should write x in terms of the ladder operators $a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$ and $a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-ip + m\omega x)$.

   (b) For what range of λ do you expect first-order perturbation theory to be valid?

   (c) What is the sign of the second-order correction to the ground state energy? Explain your reasoning clearly. (This section should be done without explicit calculation.)

3. Consider a particle of charge e that is confined to move within an infinite two-dimensional potential well given by $V(x, y) = 0$ for $0 \leq x, y \leq L$, and $V(x, y) = \infty$ otherwise. The eigenfunctions and eigenenergies are:

$$\psi_{n,m}(x, y) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right), \qquad E_{n,m} = \frac{\hbar^2\pi^2}{2mL^2}(n^2 + m^2), \quad \text{for} \quad n, m = 1, 2, 3, \ldots$$

   A small, spatially-constant electric field $\vec{E}$ is applied to the system. The corresponding perturbation to the Hamiltonian is $H' = -e\,\vec{r} \cdot \vec{E}$, where $\vec{r} = (x, y)$ is the position of the particle.

   (a) What are the values of m and n for the ground state? Calculate the first-order shift of the ground-state energy in perturbation theory.

   (b) What are the possible values of n and m for the first excited state? Calculate the matrix H'_{ab} for the perturbation in this degenerate space. Here a and b denote the quantum numbers for the degenerate states.

   (c) Calculate the corrections to the energy for the first excited state due to the perturbation. What is the "good" basis for this system?

   Potentially relevant formulae:

$$\int_0^\pi x \sin^2 x \, dx = \frac{\pi^2}{4} \qquad \int_0^{2\pi} x \sin^2 x \, dx = \pi^2 \qquad \int_0^\pi x \sin x \sin 2x \, dx = -\frac{8}{9}$$