PY 452 Final Exam

Wednesday December 16, 2009, 9:00am–noon

This is a CLOSED book exam; no calculators or electronic devices allowed. You may use a single 8.5×11 (both sides) sheet that contain your prepared handwritten notes. **PLEASE EXPLAIN YOUR WORK CLEARLY** to maximize the possibility of partial credit.

1. (5 points) Consider the infinite one-dimensional potential well $V(x)$, with $V(x) = 0$ for $0 \le x \le L$, and $V(x) = \infty$ otherwise. Suppose that 10^8 spinless Fermions of mass equal to the electron mass are contained in this well. Estimate the width of the well L so that the Fermi energy of the system is 1 eV.

2. (8 points) Use the variational approach to estimate the ground state energy of the two-dimensional harmonic oscillator,

$$H = \frac{1}{2m}\,(p_x^2 + p_y^2) + \frac{1}{2} m\omega^2\,(x^2 + y^2),$$

using the trial wavefunction $\Psi(\mathbf{r}) = A\, e^{-\alpha r}$.

Note: The Laplacian in polar coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi}\ .$$

   (a) Determine the normalization factor A.
   (b) Write the expression for $\langle \Psi | H | \Psi \rangle$ and determine the variational energy $E(\alpha)$.
   (c) In general, is $E(\alpha)$ larger or smaller than the true ground state energy? Explain why physically.
   (d) Find the optimum value of α and thereby give a bound E_{var} on the ground-state energy. Compare E_{var} with the exact ground state energy of two-dimensional harmonic oscillator.

3. (10 points) A particle is confined to an infinite two-dimensional potential well defined by the potential $V(x, y) = 0$ for $0 \le x, y \le L$, and $V(x, y) = \infty$ otherwise. The eigenfunctions and eigenenergies are:

$$\psi_{n,m}(x) = \frac{2}{L}\,\sin\left(\frac{n\pi x}{L}\right)\,\sin\left(\frac{m\pi y}{L}\right), \quad E_{n,m} = \frac{\hbar^2\pi^2}{2mL^2}(n^2 + m^2), \quad \text{for} \quad n, m = 1, 2, 3, \ldots$$

A perturbation $H' = L^2\, V_0\, \delta(x - L/4)\, \delta(y - L/4)$ is added to the Hamiltonian.

   (a) Calculate the first order shift of the ground state energy.
   (b) State the values of n and m that give the first excited state. What is the degeneracy of this state?
   (c) Write the matrix for the perturbation in this degenerate subspace and calculate the corrections to the energy of the first excited state due to the perturbation.

4. (10 points) A particle is confined to an infinite one-dimensional potential well $V(x)$, with $V(x) = 0$ for $0 \le x \le L$, and $V(x) = \infty$ otherwise. The eigenfunctions and eigenenergies are:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right), \qquad E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

A time-dependent perturbation

$$H'(t) = V'\,\frac{e^{-t^2/(2\tau^2)}}{\sqrt{2\pi\tau^2}}, \quad \text{with} \quad V' = L\, V_0\, \delta(x - L/2)$$

is added to the Hamiltonian.

(a) Calculate the matrix elements $\langle\psi_n|V'|\psi_m\rangle$.

(b) Suppose that the system is initially in the ground state ($n = 1$) at $t \to -\infty$. To first order in perturbation theory, calculate the probability that the system ends up in an eigenstate other than the ground state (*i.e.* $m \neq 1$) at $t \to \infty$.

(c) For $\tau \to 0$, how does the probability depend on the value of m? Interpret your result physically.

(d) For $\tau \to \infty$, how does the probability depend on the value of m? Interpret your result physically.

5. (7 points) In the first Born approximation, the scattering amplitude for a particle of mass m is given by

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r}')\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}\, d\mathbf{r}' ,$$

where $\mathbf{k}'$ and $\mathbf{k}$ are the incoming and outgoing wavevectors, respectively.

(a) For the delta-shell potential $V(\mathbf{r}) = \alpha\delta(r - a)$, compute $f(\theta, \phi)$ and the total cross section σ in the low-energy limit.

(b) Numerically estimate σ under the assumption that a equals the Bohr radius and that α equals the Bohr radius times 1 Rydberg and that the particle mass equals the electron mass.

(c) Repeat part (a) for the case of arbitrary energy. Here you need to keep the exponential factor in the integral given above for f. Show that for $\mathbf{k}' - \mathbf{k} \to 0$, you recover your result in part (a).