Quantum Annealing and Spin Glasses

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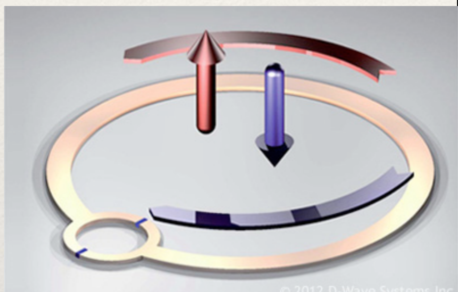
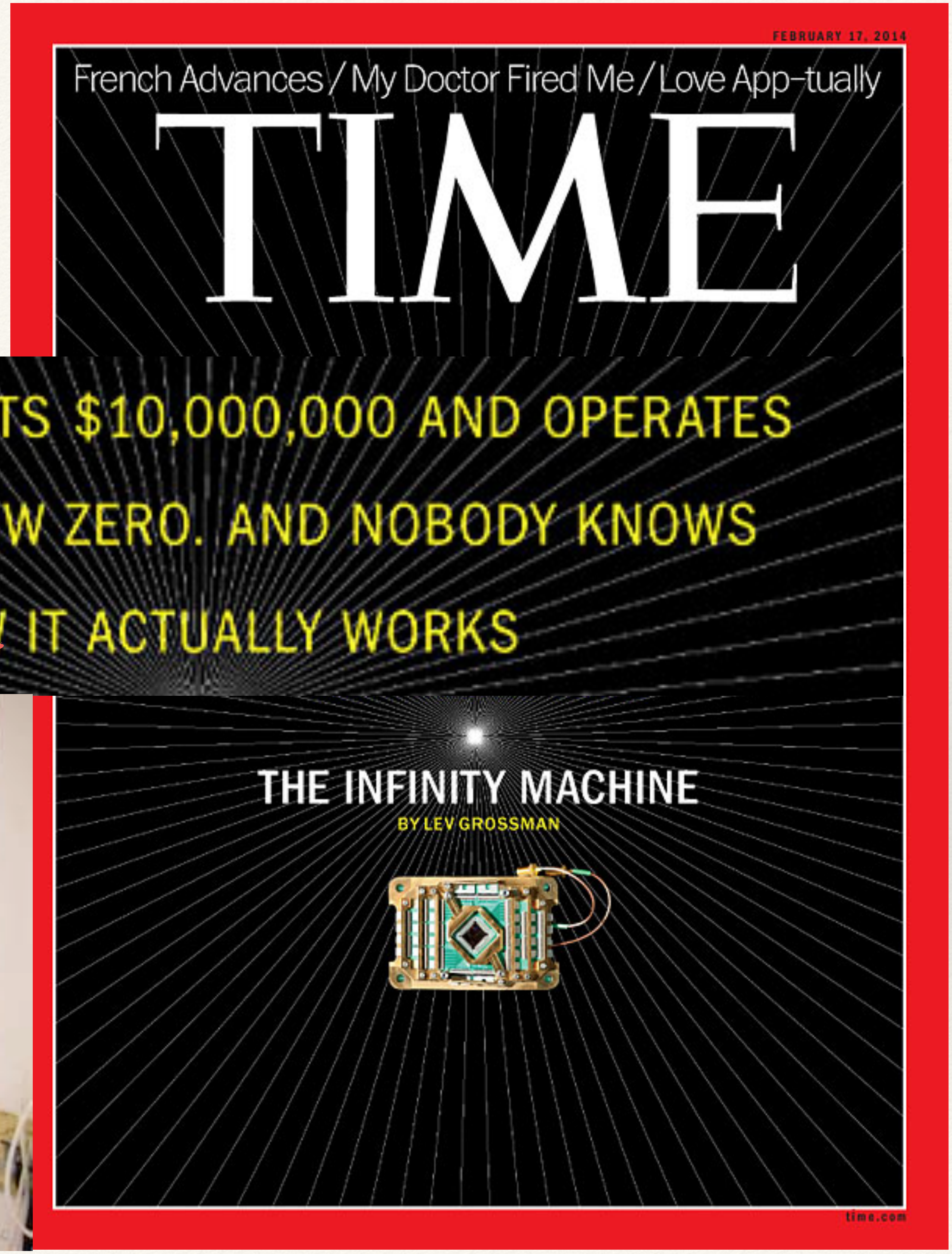
- The D-wave quantum annealer
- Principles of quantum annealing
- Critical (Kibble-Zurek) scaling out of equilibrium
- D-wave experiments; critical scaling in a quantum spin glass

The D-Wave quantum annealer

Claim: Can solve hard optimization problems using quantum mechanics





Classical optimization; a physics analogy

Minimize a 'cost function' dependent on many parameters

- Cost function $\Longleftrightarrow$ energy $E(X)$
- Parameters $\Longleftrightarrow$ configuration space, $X=\{x_1,x_1,\dots,x_N\}$

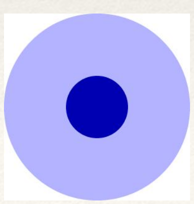
Statistical Physics:

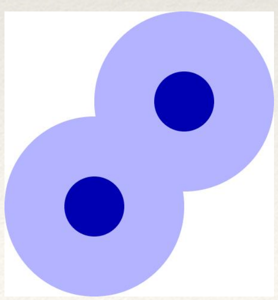
$$\langle E\rangle = \frac{1}{Z}\int dX\, E(X)\mathrm{e}^{-E(X)/T}$$

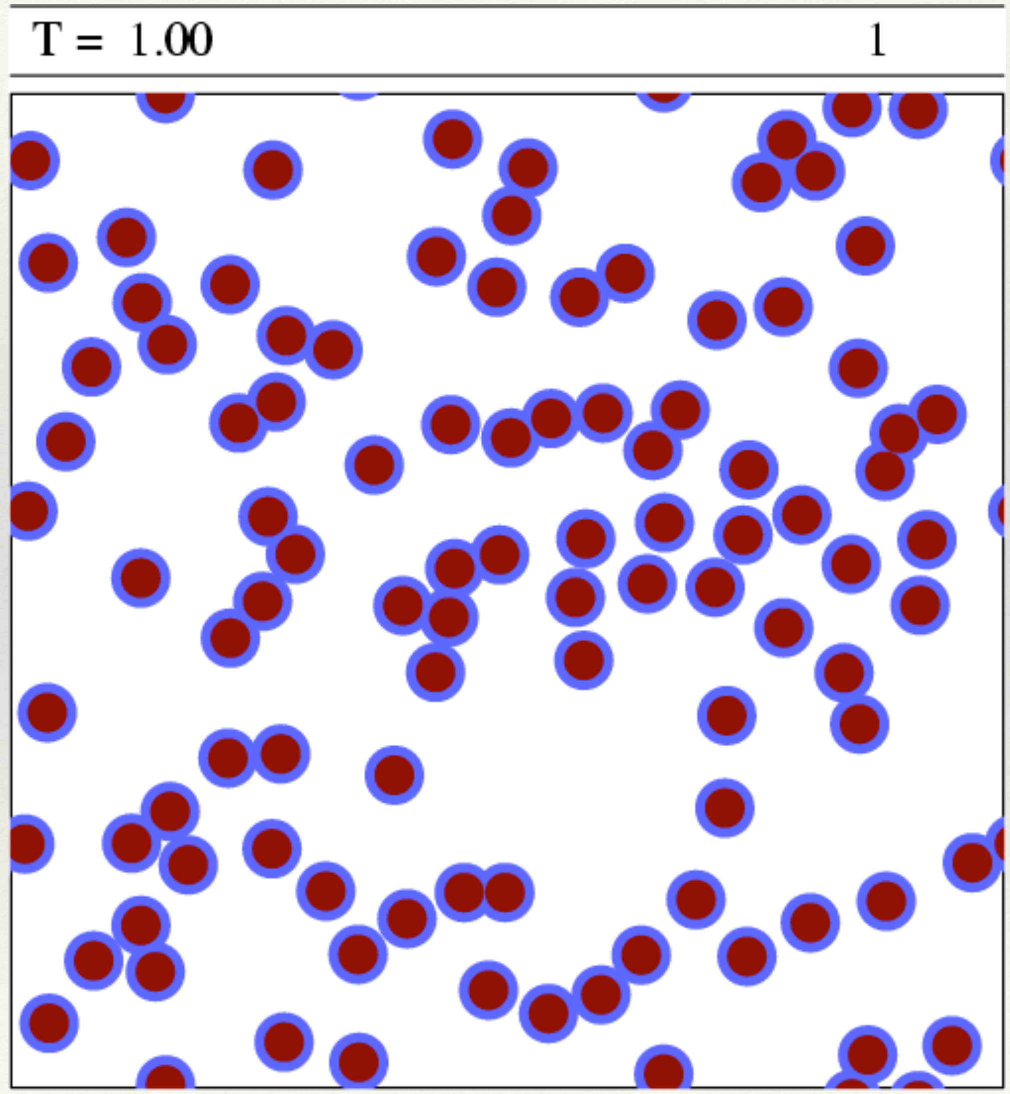
$$\lim_{T\to 0}\langle E(T)\rangle = E_0$$

Example: MC simulations

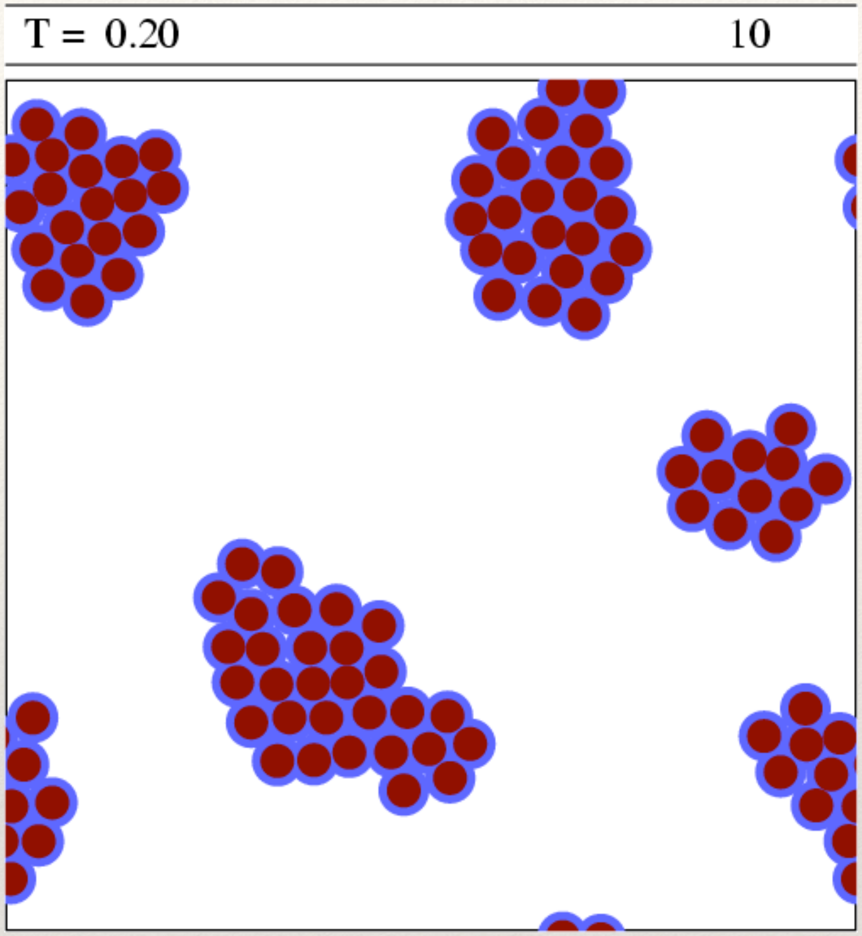
$$V(r) = \begin{cases} \infty, & r <\leq r_1 \\ -V, & r_1 < r \leq r_2 \\ 0, r > r_2 & \end{cases}$$

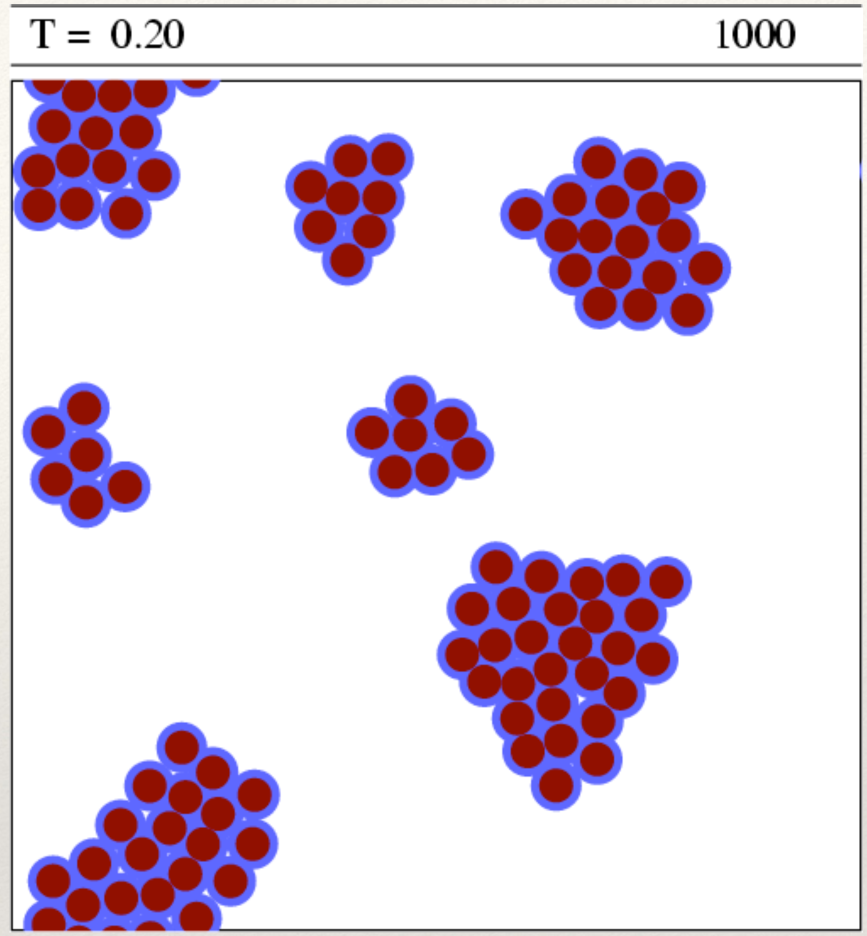






Let’s lower the temperature….





Transition into liquid state has taken place

Slow movement & growth of droplets

- simulation is not strictly equilibrated

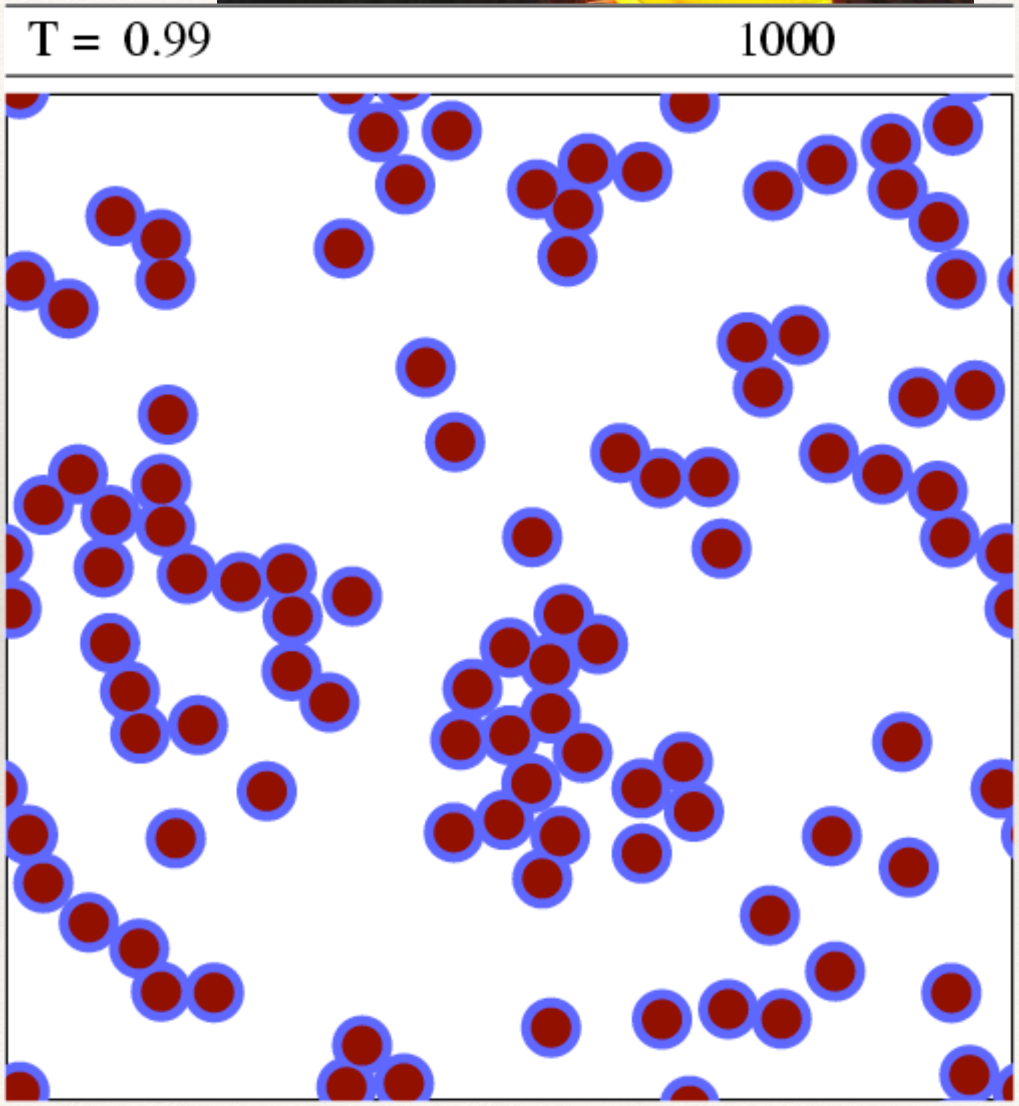
Is there a better way to reach equilibrium at low T?

Annealing a metal:

Removal of crystal defects by heating followed by slow cooling

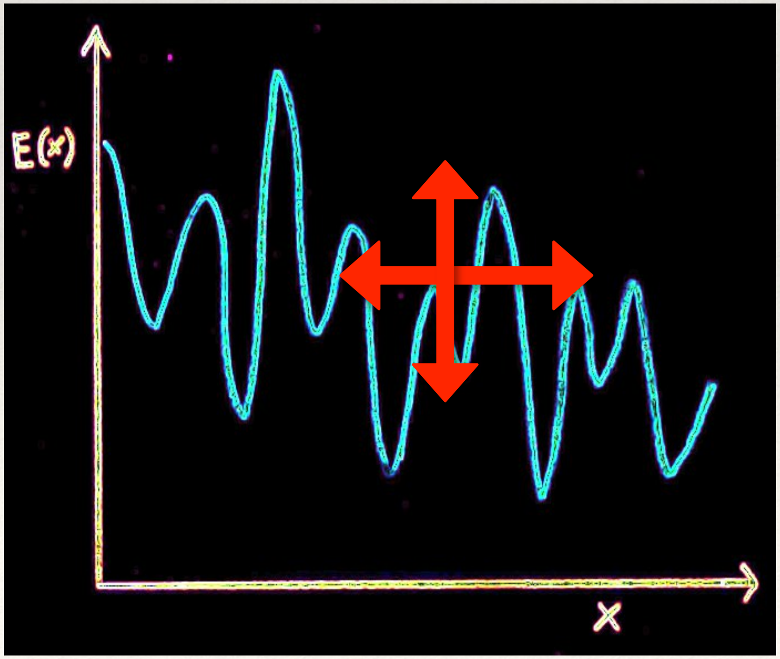
Simulated Annealing:

MC with slowly decreasing T
- Kirkpatric, Gelatt, Vecci, 1983



Quantum mechanics analogy - quantum annealing

Thermal fluctuations
Quantum tunneling



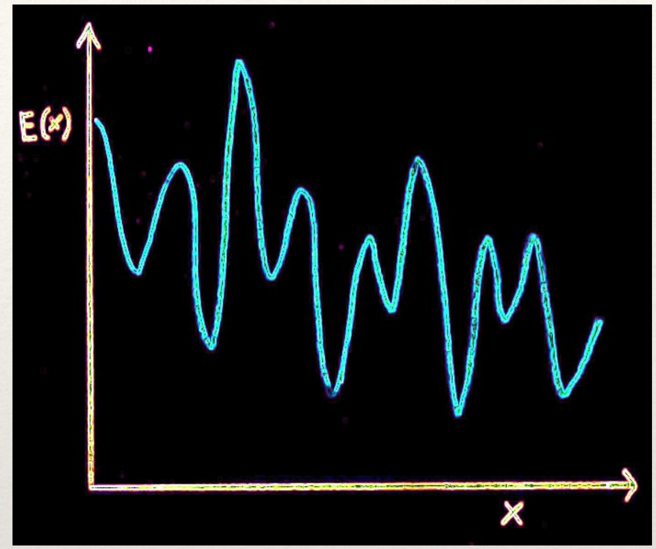
Quantum Annealing

Reduce quantum fluctuations as a function of time
- start with a simple quantum mechanical Hamiltonian
- end with a complicated classical Hamiltonian (potential)

$$H(s) = sH_{\text{classical}} + (1-s)H_{\text{quantum}}$$

$$s(t) = vt, \quad v = 1/t_{\max}$$

$$[H_{\text{quantum}}, H_{\text{classical}}] \neq 0$$



Single-particle example (not practical)

$$H_{\text{classical}} = V(x) \quad H_{\text{quantum}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

- annealing would correspond to m → m/(1-s) (→ ∞ when s → 1)

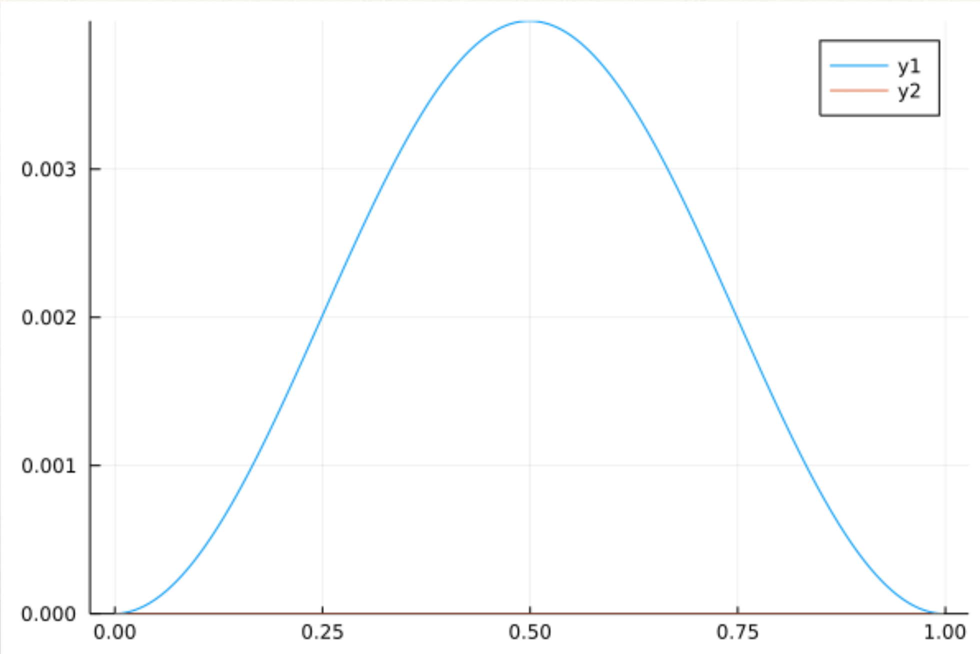
Adiabatic Theorem:

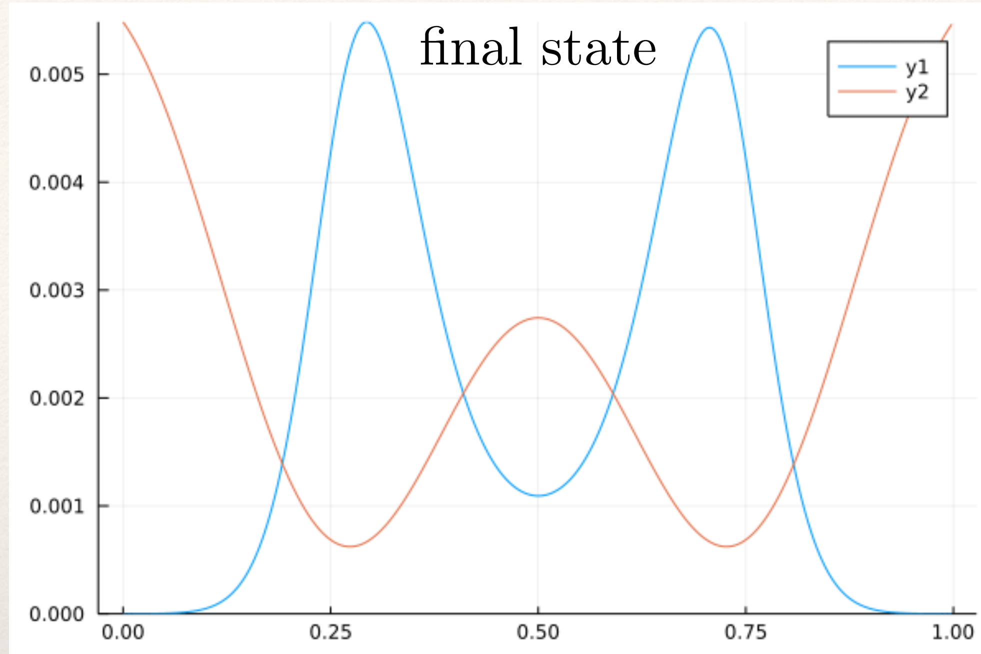
If the annealing velocity v is small enough
the system stays in the ground state of H[s(t)] at all times

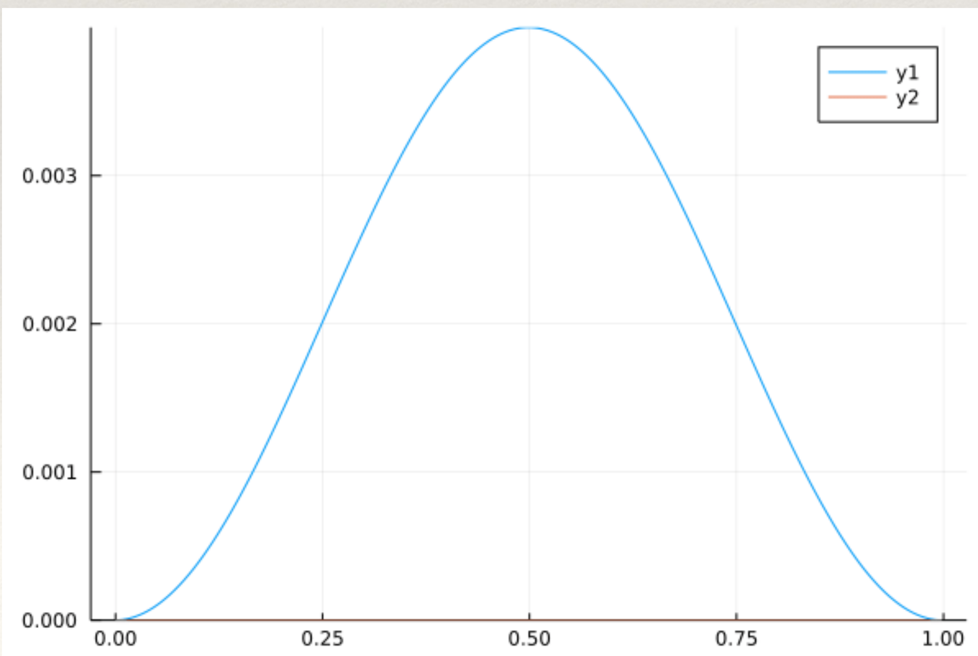
At $t=t_{max}$ we then know the minimum of V(x): $\Psi(x) = \delta(x - x_0)$

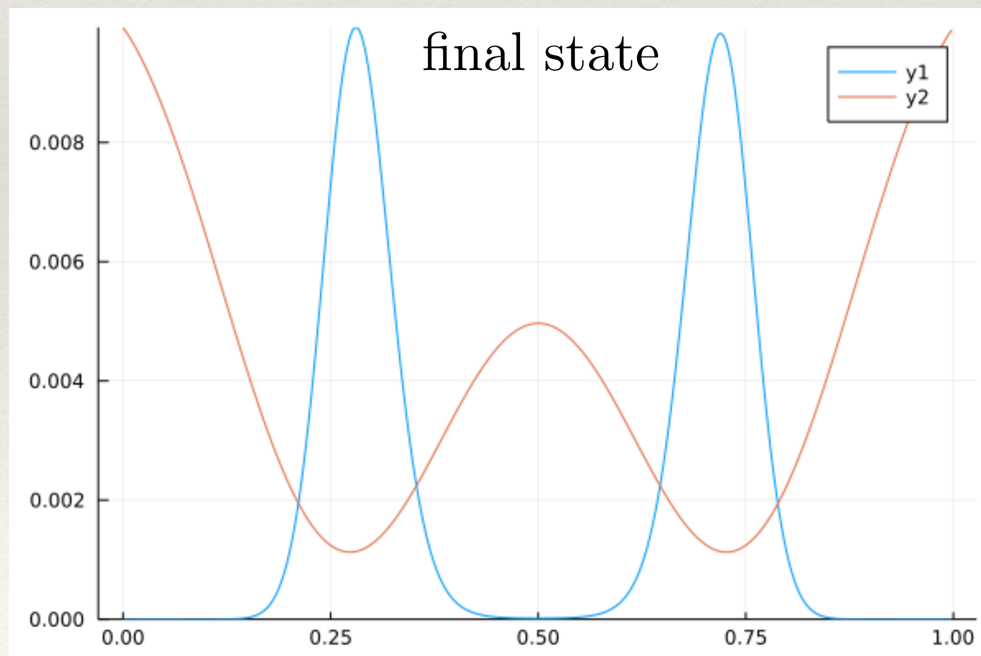
- if slow enough annealing
- deviations for finite annealing time

Particle in a box: mass 1 → ∞, potential 0 → 1









Can quantum annealing be more efficient than thermal annealing?

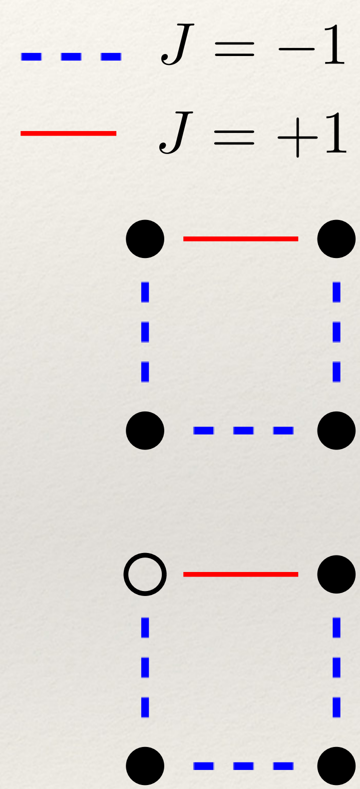
- when implemented as a device

Useful paradigm for quantum computing?

- Kadowaki, Nishimori 1996, Farhi et al. 2001,...

Spin glasses: Ising models with random frustrated interactions

$$H = \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\sigma_i^z\sigma_j^z, \quad \sigma_i^z \in \{-1,+1\}$$



Hard to find ground states if the interactions are highly frustrated (spin glass phase)
- many states with same or almost same energy

T>0 Glass phase: $\langle\sigma_i\rangle \neq 0$, spins “frozen” in time
- many different such thermodynamic (stable) states

Many optimization problems can be mapped onto some general Ising model
- hard problems correspond to spin glass physics

Introduce quantum fluctuations; **quantum spin glasses**

$$H_{\text{quantum}} = -h\sum_{i=1}^{N}\sigma_i^x = -h\sum_{i=1}^{N}(\sigma_i^+ + \sigma_i^-)$$

Annealing through a quantum phase transition

Many quantum phase transitions have a gap minimum at the transition point
- scaling in system size (length) L

$$\Delta(L) \propto L^{-z} \quad \text{(continuous transition)}$$

$$\Delta(L) \propto \mathrm{e}^{-aL} \quad \text{(first-order transition)}$$

Suggesting QA efficient if first-order transition can be avoided

Relaxation time: $\tau \sim \xi^z \sim \Delta^{-z} \;\to\; L^z$

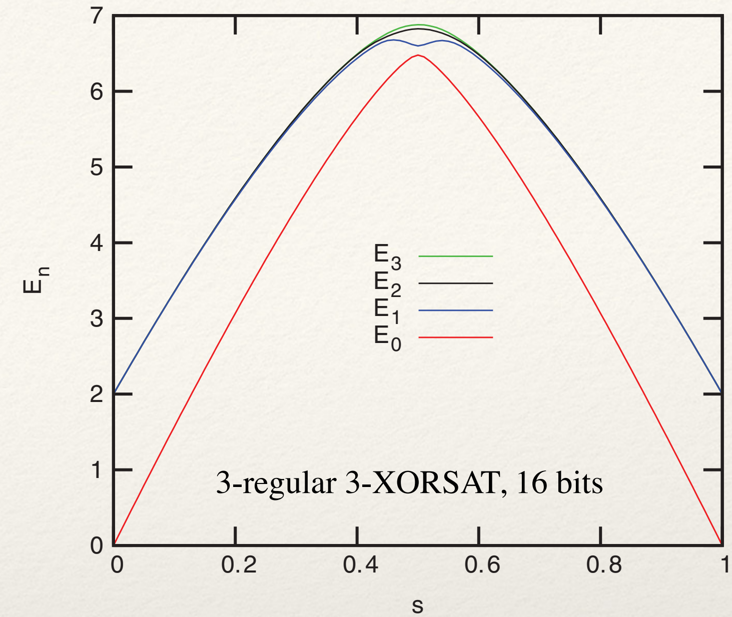


Fig from Farhi, Gossett, Hen, Sandvik, Shor, Young. Zamponi, PRA 2012

Kibble-Zurek mechanism

- time remaining to transition ~ τ →
- quasi-adiabatic to critical point if $t_{anneal} \sim L^{(z+1/\nu)}$

$$\xi_v \propto v^{-\nu/(z\nu+1)} \quad (\text{if} \ll \mathrm{L})$$

Classical version:
Kibble 1976
Zurek 1985

Quantum case:
Polkovnikov, 2005
Dziarmaga, 2005
Zurek, Dorner, Zoller, 2005

Other dynamical effects inside ordered phases (e.g., quantum coarsening)

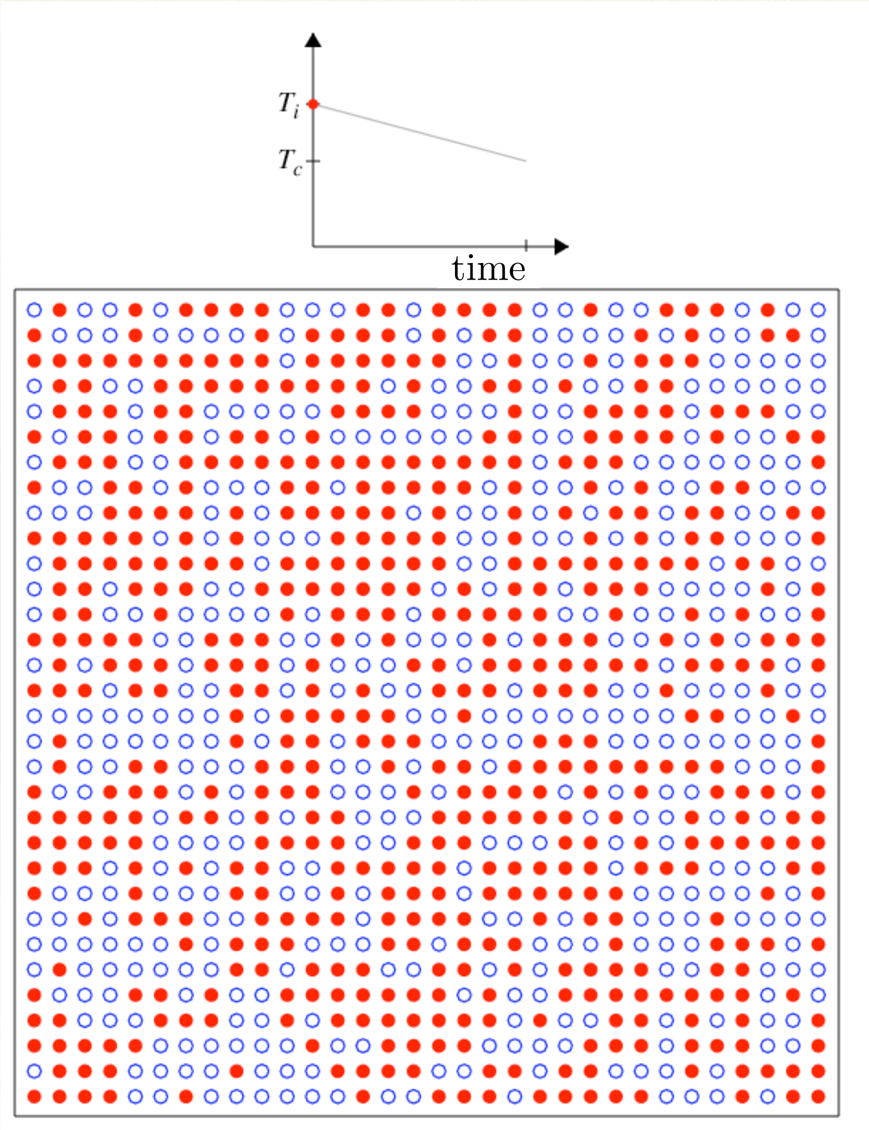
Glass-type phases have avoided level crossing, small gaps
- QA to classical ground state likely scales exponentially

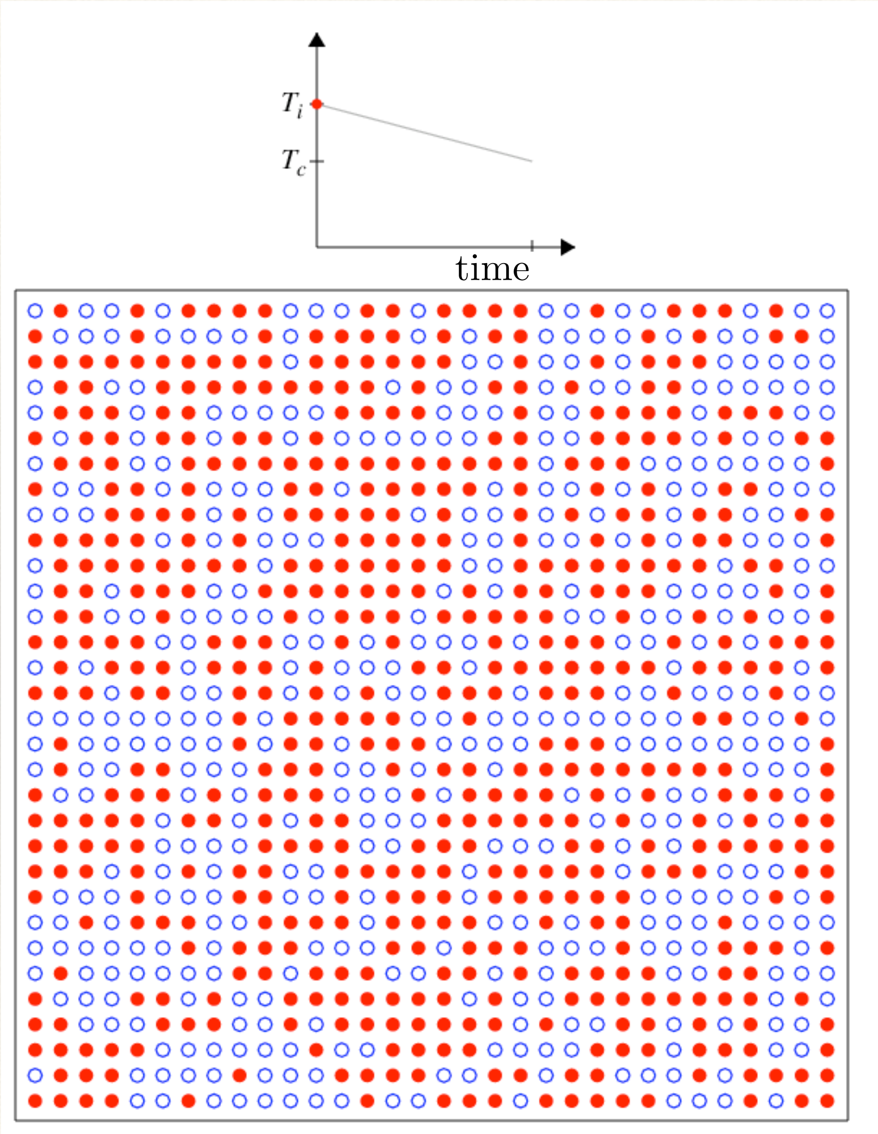
Classical simulated annealing, Kibble-Zurek scaling

- anneal from high T to Tc: time needed to stay in equilibrium?

$$t_{\mathrm{KZ}} \sim L^{z+1/\nu}$$

Animations by Cheng-Wei Liu





Equilibrium finite-size scaling

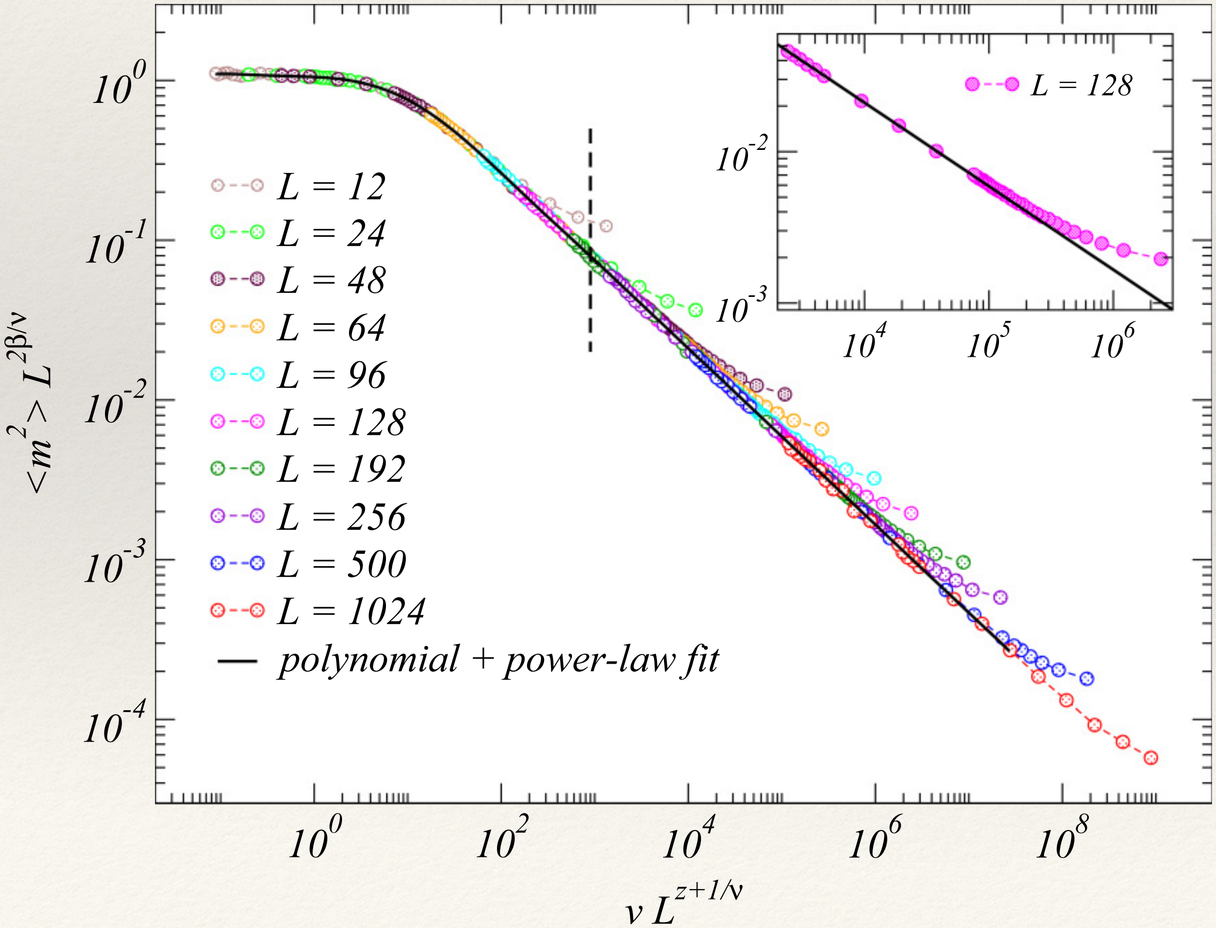
$$A \sim \delta^{\kappa} \to A(L,\delta) = L^{-\kappa/\nu} f(\delta L^{1/\nu}), \quad \delta = T - Tc$$

KZ mechanism → generalized finite-size scaling

$$A(L,\delta,v) = L^{-\kappa/\nu} f(\delta L^{1/\nu}, vL^{z+1/\nu}), \quad v = 1/t_{\mathrm{anneal}}$$

Simplest case: stop at critical point δ=0, $A = L^{-\kappa/\nu} f(vL^{z+1/\nu})$

C.-W. Liu, A. Polkovnikov, A. W. Sandvik, PRB 2014



2D Ising

$\beta = 1/8, \nu = 1$

Find dynamic exponent z by optimizing data collapse

$z \approx 2.17$

Quantum (transverse field) Ising model (2+1 dim)

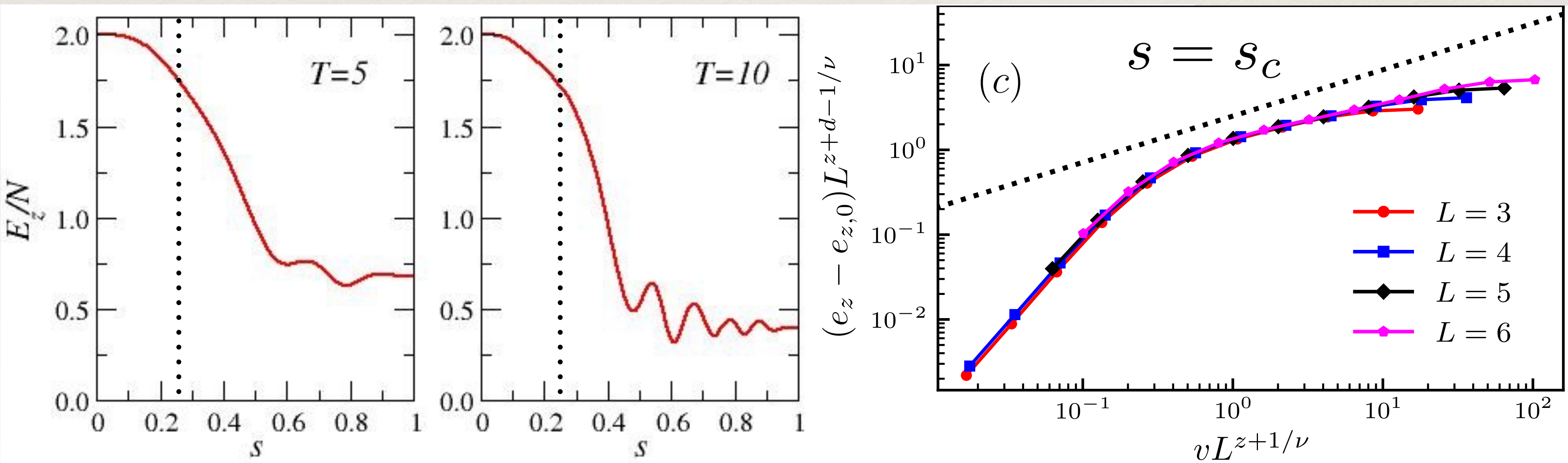
$$H = -s\sum_{\langle i,j\rangle}\sigma_i^z\sigma_j^z - (1-s)\sum_i \sigma_i^x \qquad L\times L \text{ lattice}$$

Adiabatic limit: Quantum phase transition at s ≈ 0.25 (L→∞)
- finite-size scaling at the critical point

Annealing time *T* (velocity *v=1/T*), scaling in L and v

Kibble-Zurek: quasi-adiabatic if v ~ $L^{-(z+1/\nu)}$ $\quad z = 1, \nu = 0.63$

Excess Ising energy, numerical exact time evolution, L=4



Long-time limit in the ordered phase reflects other dynamical processes
- coarsening, domain walls,…

Weinberg, Xu, Sandvik (unpublished)

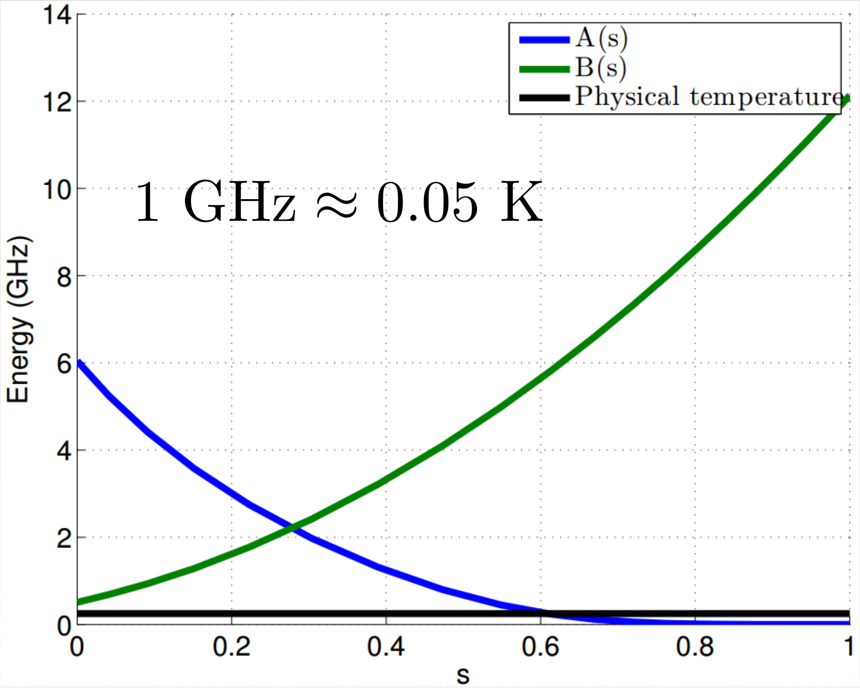
D-wave Hamiltonian: Ising model

$$H_{\text{classical}} = \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} \sigma_i^z \sigma_j^z, \quad \sigma_i^z \in \{-1, +1\}$$

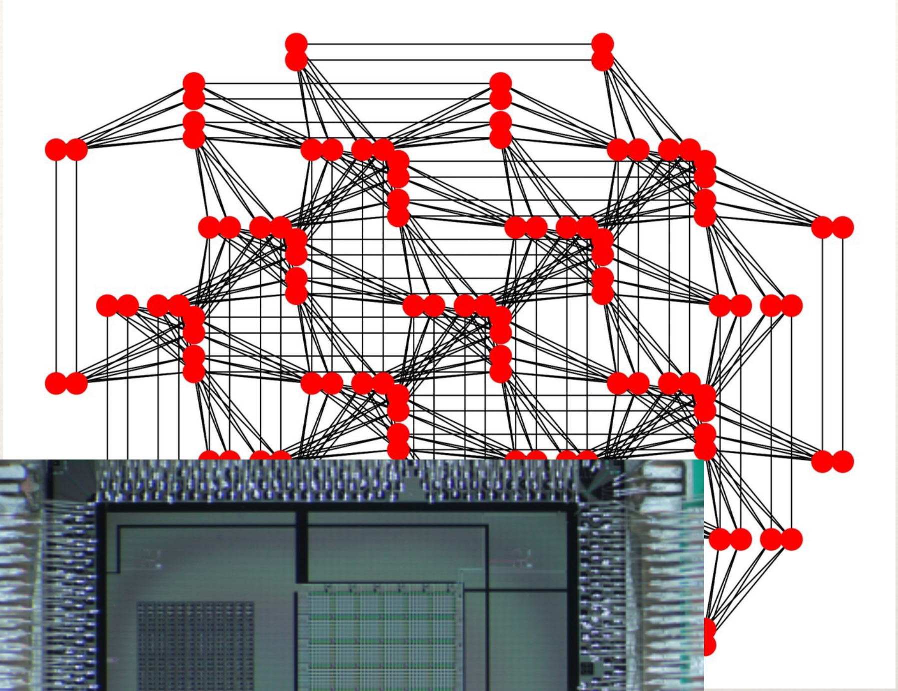
Interactions J_{ij} are programmable
- restricted, depending on the device

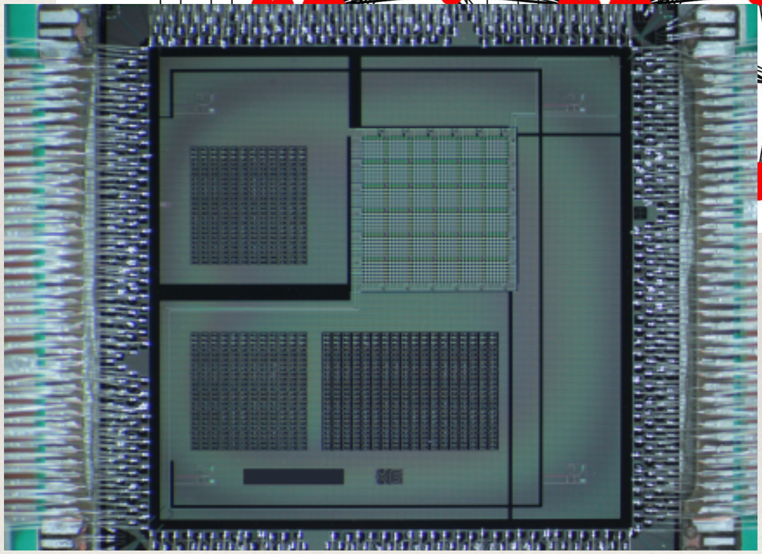
Quantum fluctuations; transverse field

$$H_{\text{quantum}} = -\sum_{i=1}^{N} \sigma_i^x = -\sum_{i=1}^{N} (\sigma_i^+ + \sigma_i^-)$$



Newest D-Wave architecture:
Pegasus (2021), > 5000 qubits





$$H(t) = A(s[t])H_{\text{quantum}} + B(s[t])H_{\text{classical}}$$

$s[t] \in [0,1]$ is a programmable sequence of increasing linear functions of t

3D Ising spin glass

$$E = \sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j$$

<i,j> nearest neighbors on 3D cubic lattice

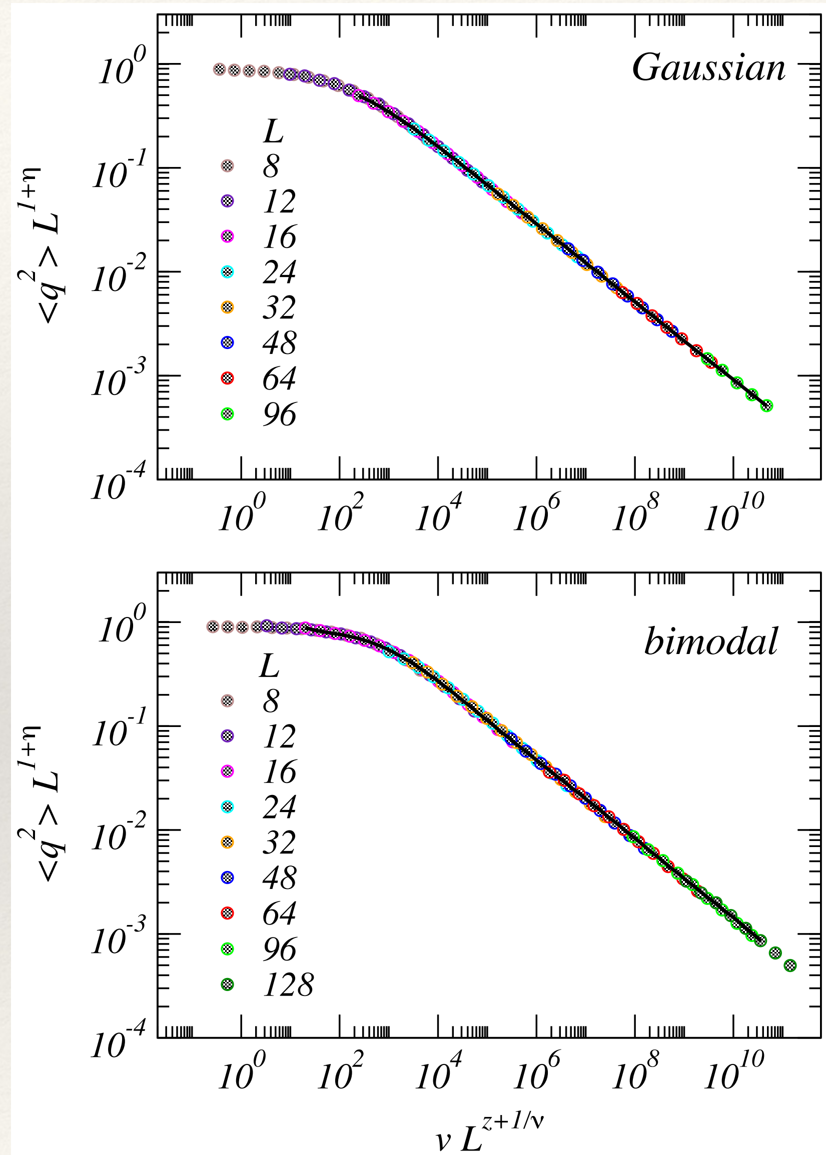
J_{ij} random, mixed signs (e.g.,bimodal ±1)

Finding ground state(s) is a prototypical hard optimization problem; time ~ exponential in $N=L^3$

T > 0 transition to glass state
- T_c and critical exponents known

Edwards-Anderson order parameter
- Two independent simulations with same random coupling realization (replicas)

$$q = \sum_{i=1}^{N} \sigma_i^{(1)}\sigma_i^{(2)}$$



What is the dynamic exponent z?
Anneal to T_c and do same kind of KZ analysis as before → **z ≈ 6.0**

C.-W. Liu, A. Polkovnikov, A. W. Sandvik, A. P. Young, PRE 2015

Nature 617, 61 (2023)

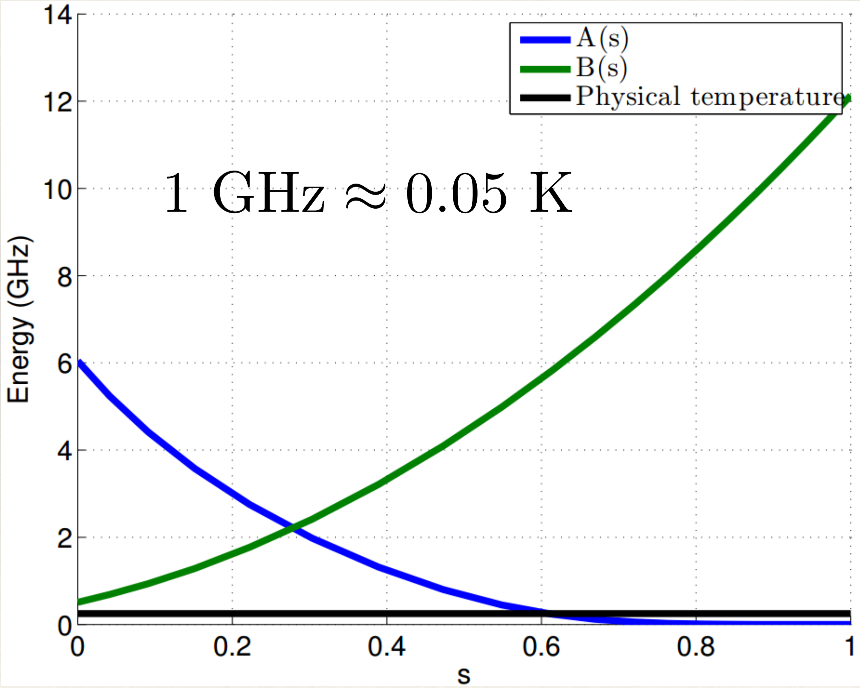
Quantum critical dynamics in a 5,000-qubit programmable spin glass

Andrew D. King[1] ✉, Jack Raymond[1], Trevor Lanting[1], Richard Harris[1], Alex Zucca[1], Fabio Altomare[1], Andrew J. Berkley[1], Kelly Boothby[1], Sara Ejtemaee[1], Colin Enderud[1], Emile Hoskinson[1], Shuiyuan Huang[1], Eric Ladizinsky[1], Allison J. R. MacDonald[1], Gaelen Marsden[1], Reza Molavi[1], Travis Oh[1], Gabriel Poulin-Lamarre[1], Mauricio Reis[1], Chris Rich[1], Yuki Sato[1], Nicholas Tsai[1], Mark Volkmann[1], Jed D. Whittaker[1], Jason Yao[1], Anders W. Sandvik[2] ✉ & Mohammad H. Amin[1,3] ✉

$$H(t) = -A[s(t)]\sum_{i=1}^{N}\sigma_i^x + B[s(t)]\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z$$

$$s(t=0)=0$$

$$s(t=t_{\mathrm{anneal}})=1$$



The qubits at t=0 are prepared to

$$\sigma_1^x = 1, \quad |\psi\rangle_i = |\uparrow_z + \downarrow_z\rangle_i$$

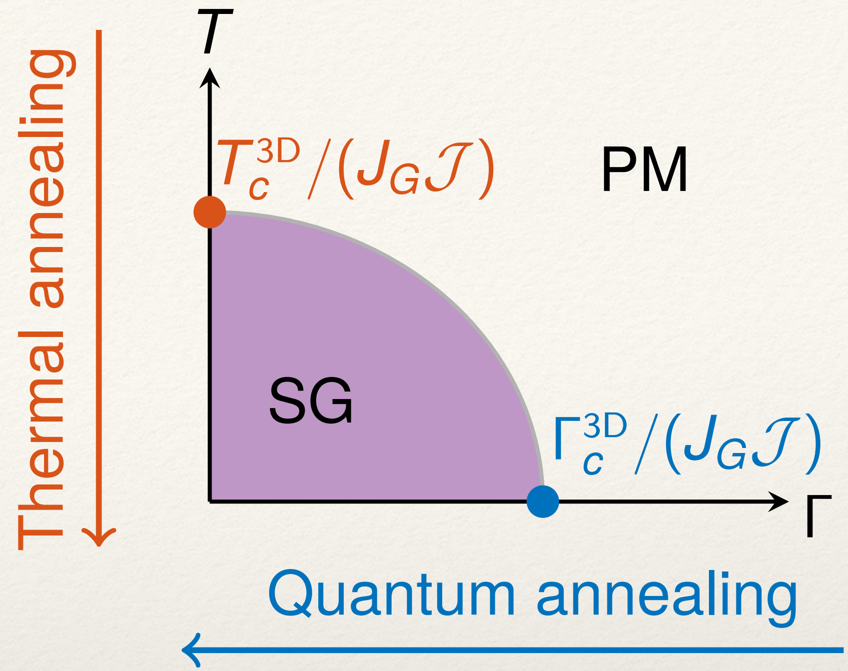
The spin-z states are recorded at t=t_{anneal} (i.e, at s=1, H in classical limit)

Equilibrium Phase Diagram

$$H(t) = -\Gamma \sum_{i=1}^{N} \sigma_i^x + \sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z$$

Can be studied with quantum Monte Carlo simulations
- Rieger & Young, PRL 1994 + ...

T=0 critical exponents known, including z (from equilibrium imaginary-time correlations)



Thermal critical point

$$\nu \approx 2.6, \;\; z \approx 6.0$$

Quantum critical point

$$\nu \approx 0.65, \;\; z \approx 1.3$$

To observe criticality in simulated or quantum annealing:

$$t_{\text{anneal}} \propto L^{z+1/\nu} \quad (N = L^3 \text{ spins})$$

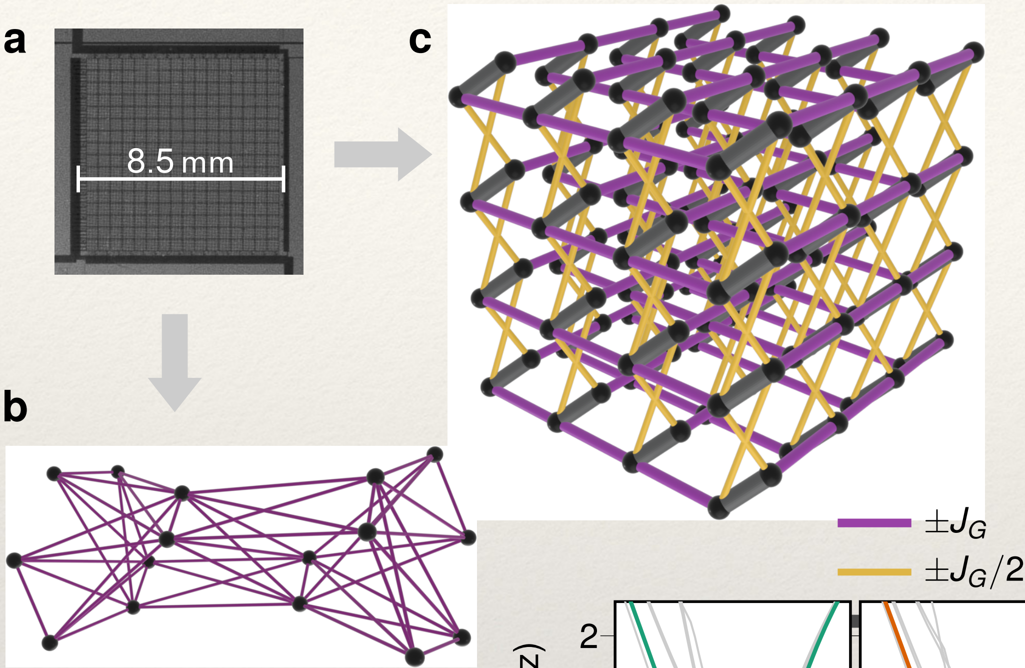
Much shorter time scale for given L in quantum annealing!

Complications in D-wave experiments:
- spin state only accessed at Γ=0, T low but >0 (environment)

Slow dynamics in glass state: **remnants of criticality in final state?**

Two tests of quantum dynamics

3D connectivity on the D-wave Advantage QPU (Pegasus connectivity) - requires dimer geometry; two “hard coupled” qubits is a logical qubit

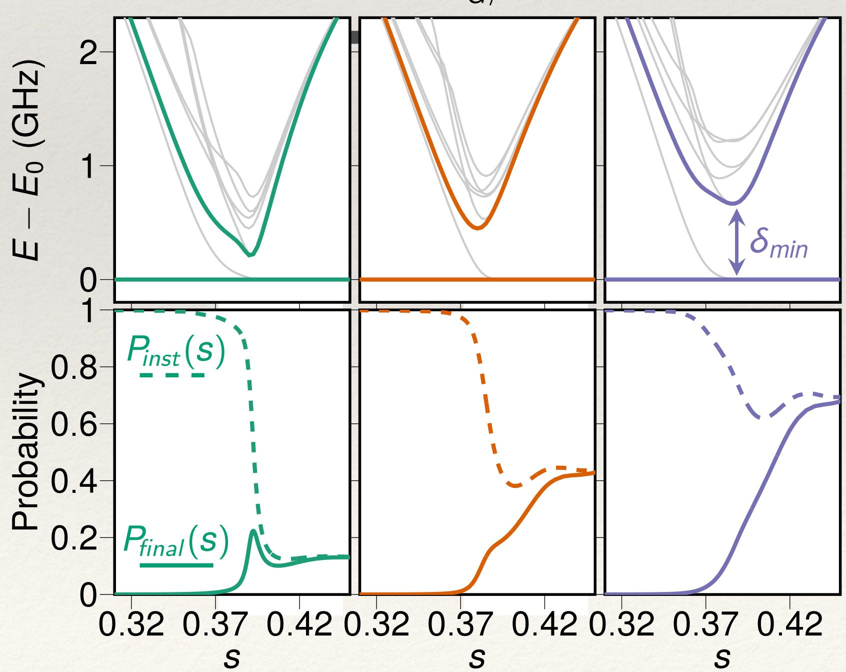


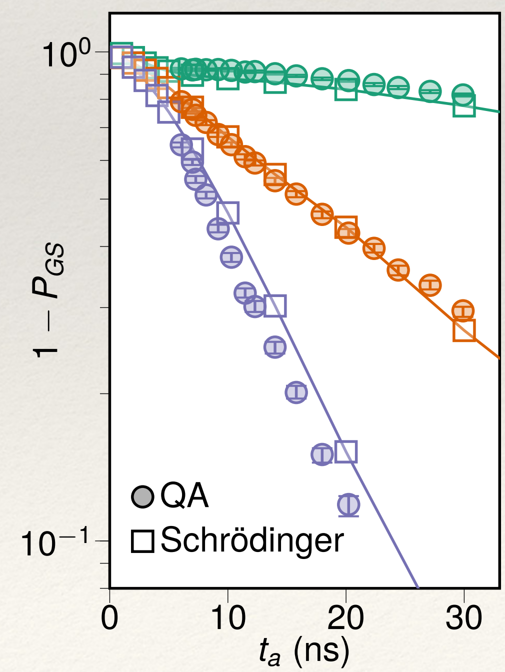
$$N = 2 \times L^3$$

KZ finite-size scaling for L up to 12

Critical exponents?

16 spins (8 dimers) compare with exact Schrödinger dynamics





Scaling with system size and annealing rate ($v=1/t_{anneal}$)

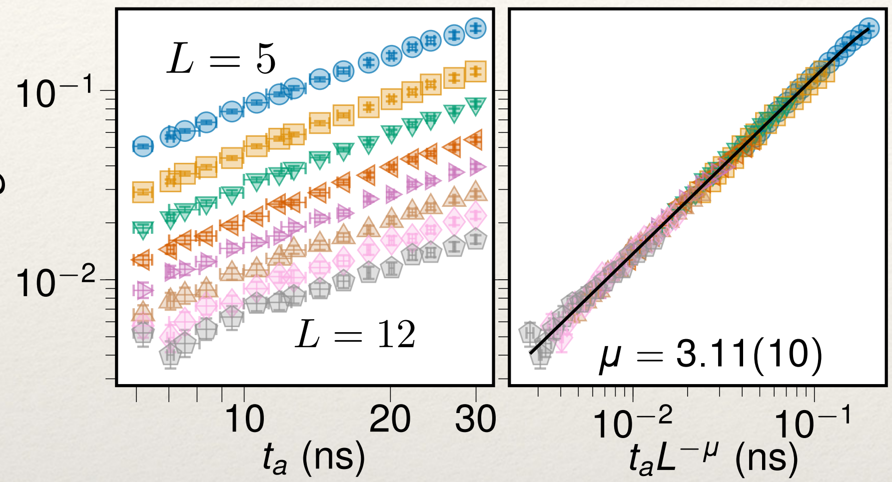
First consider a dimensionless quantity; the Binder cumulant

$$U = \frac{1}{2}\left(3 - \frac{\langle q^4 \rangle}{\langle q^2 \rangle^2}\right)$$

expected for near-critical systems

$$U \propto f(vL^{\mu}),\;\; \mu = z + 1/\nu$$

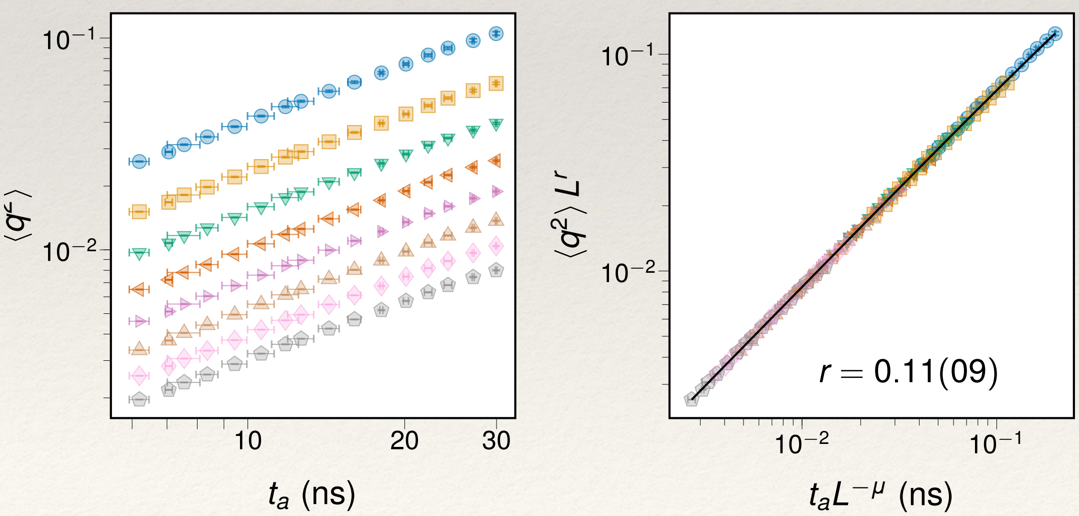
c



$\mu \approx 2.9$ expected

The expected critical scaling is closely reproduced!

Squared EA order parameter



Same value of μ as above

r ~ 1 expected at criticality

r ~ 0 from the fact that the system evolves slightly inside the glass phase

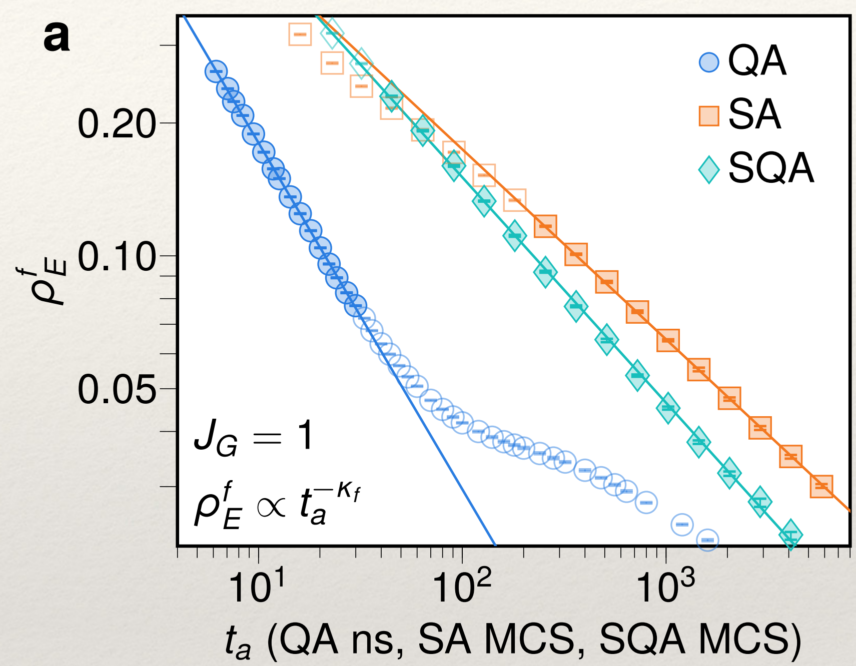
Summary

The D-wave quantum annealer realizes true quantum dynamics

- successful emulation of 3D Ising spin glass

Is there quantum speedup?

Largest lattice considered 2×15×15×12 qubits



- the excess energy density decays faster with annealing time than in classical simulated annealing
- until performance impacted by "noise" after ~50 ns

Longer coherence times and more qubits expected in the near future

- real prospects of addressing problems inaccessible to classical calculations
- access to the spins at any point in the evolution would be desirable for physics emulation

Recent D-Wave paper with more examples: Science 388, 199 (2025)