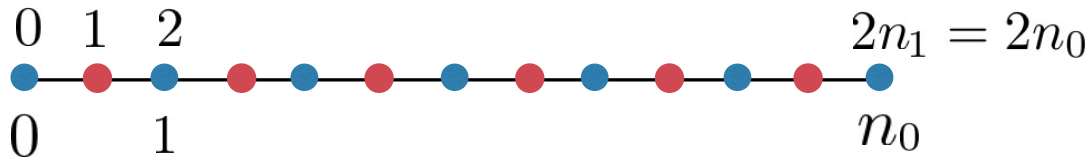


Romberg integration

Idea: Use two or more trapezoidal integral estimates, extrapolate

- step sizes (decreasing order) h_0, h_1, \dots, h_m , integral estimates I_0, I_1, \dots, I_m
- use polynomial of order n to fit and extrapolate to $h=0$
- error for given h scales as h^2 (+ higher even powers only)
- use polynomial $P(x)$ with $x=h^2$

Simplest case: 2 points ($m=1$), using $h_0=(b-a)/n_0$ and $h_1=h_0/2$ ($x_1=x_0/4$)



Function evaluation once only for each point needed

$$I_0 = I_\infty + \epsilon x_0, \quad I_1 = I_\infty + \epsilon x_0/4$$

$$\rightarrow I_\infty = \frac{4}{3}I_1 - \frac{1}{3}I_0 + O(h_0^4) \quad [O(x_0^2)]$$

reducing h by 50%

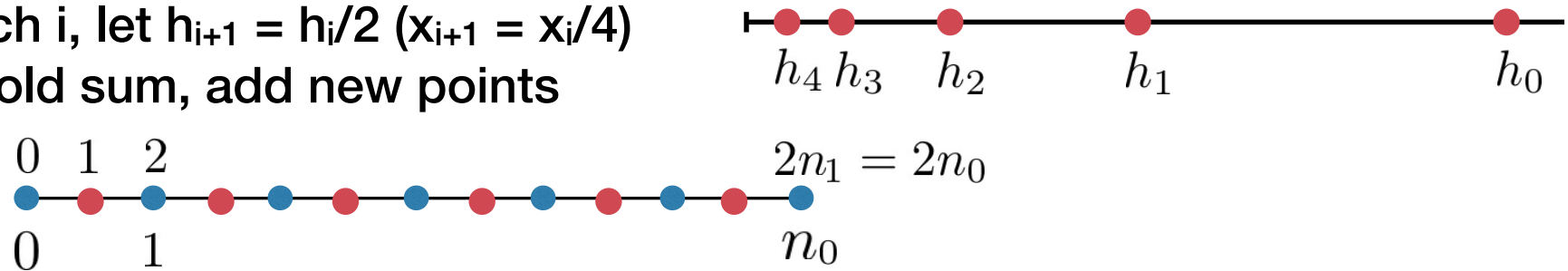
- error should be 1/4 of previous
- ϵ is unknown factor, eliminated

Computation cost doubled, error reduced by two powers of h_0 !

General case; $h_0, h_1, \dots, h_m \rightarrow l_1, l_2, \dots, l_m$

For each i , let $h_{i+1} = h_i/2$ ($x_{i+1} = x_i/4$)

- save old sum, add new points



How to construct a polynomial of order n going through $n+1$ point pairs (x_i, y_i)

$$P(x) = \sum_{i=0}^m y_i \prod_{k \neq i} \frac{x - x_k}{x_i - x_k} \quad \mathbf{x_i = h_i^2}$$

Evaluate (this is the extrapolation) at $x=0$ ($h=0$)

$$I_\infty = \sum_{i=0}^m I_i \prod_{k \neq i} \frac{-h_0^2 2^{-2k}}{h_0^2 (2^{-2i} - 2^{-2k})} = (-1)^m \sum_{i=0}^m I_i \prod_{k \neq i} \frac{1}{2^{2(k-i)} - 1}$$

Error decreases very rapidly: $O(h^{2(m+1)})$

Implemented in [romberg.jl](#) (both closed and open cases)