

Simple Harmonic Oscillations and Resonance

Spring Simple Harmonic Oscillator

We have an object attached to a spring. The object is on a horizontal frictionless surface. We move the object so the spring is stretched, and then we release it. The object oscillates back and forth in what we call **simple harmonic motion**, in which no energy is lost.

Spring constant

To be able to describe the oscillatory motion, we need to know some properties of the spring. One key property is that if the length of the spring is shortened or lengthened by an amount Δl from its equilibrium value, the spring experiences a restoring force proportional to Δl . We call it a restoring force because it always acts in a direction to return the length of the spring to the equilibrium value.

$F = -k\Delta l$

k is the spring constant

Potential Energy stored in a Spring

For a spring that is stretched or compressed by an amount Δl from the equilibrium length, there is potential energy, U , stored in the spring:

$$U = \frac{1}{2} k(\Delta l)^2$$

In a simple harmonic motion, as the spring changes length (and hence Δl), the potential energy changes accordingly.

Understanding oscillations

We have an object attached to a spring. The object is on a horizontal frictionless surface. We move the object so the spring is stretched, and then we release it. The object oscillates back and forth in what we call simple harmonic motion, in which no energy is lost. How do we find the object's maximum speed?

Understanding oscillations

We use energy conservation.

$$U_{\max} = K_{\max} \quad \Rightarrow \quad \frac{1}{2}kA^2 = \frac{1}{2}mv_{\max}^2$$

A is called the amplitude – the maximum distance from equilibrium. The key lesson to take away from this is that you already know a lot about analyzing simple harmonic motion situations- namely apply energy conservation, especially when you want to relate a speed to a position.

Splitting the energy

An object attached to a spring is pulled a distance A from the equilibrium position and released from rest. It then experiences simple harmonic motion. When the object is $A/2$ from the equilibrium position how is the energy divided between spring potential energy and the kinetic energy of the object? Assume mechanical energy is conserved.

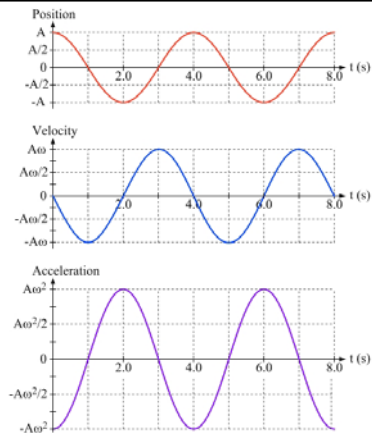
1. The energy is 25% spring potential energy and 75% kinetic.
2. The energy is 50% spring potential energy and 50% kinetic.
3. The energy is 75% spring potential energy and 25% kinetic.
4. One of the above, but it depends whether the object is moving toward or away from the equilibrium position.

Splitting the energy

The total energy of the SHM in the spring, E_{tot} , is equal to the potential energy of the spring when it is maximally stretched. It is because in that case there is no kinetic energy and the entire mechanical energy comes from U . So we have, $E_{\text{tot}} = \frac{1}{2} kA^2$. When the object is at $x = A/2$, $U = \frac{1}{2} k(A/2)^2 = E_{\text{tot}}/4$. Since the mechanical energy is conserved, the kinetic energy at $x = A/2$ is given by, $K = E_{\text{tot}} - U = 3E_{\text{tot}}/4$.

Motion Graphs

If we graph position, velocity, and acceleration of the object on the spring, as a function of time, we get the plots shown at right. The example used here has the period of oscillations equal to 4.0 seconds.



Equations

$$x = A \cos(\omega t)$$

In general,
 $x = A \cos(\omega t + \theta_0)$

$$v = -A\omega \sin(\omega t)$$

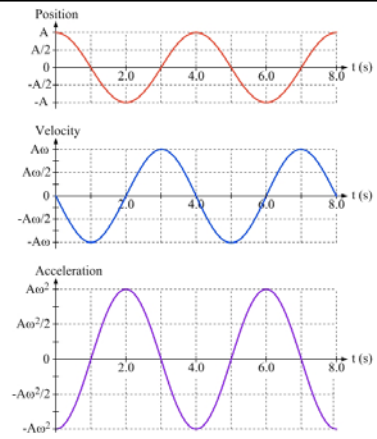
$$v_{\text{max}} = A\omega$$

$$a = -A\omega^2 \cos(\omega t)$$

Or, $a = -\omega^2 x$

$$a_{\text{max}} = A\omega^2$$

ω is called the **angular frequency**.



What determines angular frequency?

In general, we have $\vec{a} = -\omega^2 \vec{x}$.

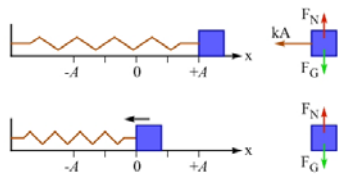
In any specific case, the angular frequency is determined by the forces involved. For an object on a spring, we have:

$$\Sigma \vec{F} = m\vec{a}$$

$$-k\vec{x} = m\vec{a}$$

$$\vec{a} = -\frac{k}{m} \vec{x}$$

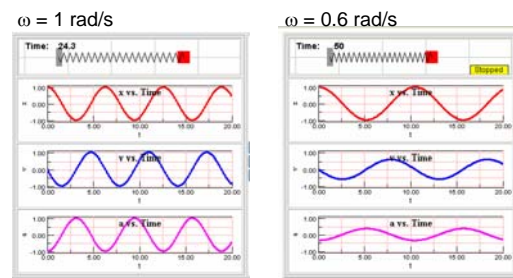
$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$



A change in frequency

Simulation

The first set of graphs (x-t, v-t, a-t) is for an angular frequency $\omega = 1$ rad/s. The second set of graphs is for $\omega = 0.6$ rad/s.



A change in frequency

The first set of graphs is for an angular frequency $\omega = 1$ rad/s. The second set of graphs is for $\omega = 0.6$ rad/s. This change of ω is accomplished either by decreasing the spring constant or by increasing the mass. Which change did we make in this case?

1. We decreased the spring constant
2. We increased the mass
3. We could have done one or the other, you can't tell the difference

A change in frequency

We can't tell the difference. All we can tell is that the angular frequency has changed. Notice that besides the period oscillation, we can also perceive that ω is changed from the fact that the maximum velocity ($= A\omega$), and the maximum acceleration ($= A\omega^2$) is changed, while A is not changed.

The graphs give us no information about whether the spring constant or the mass is different.

Understanding Oscillations from Energy Graphs

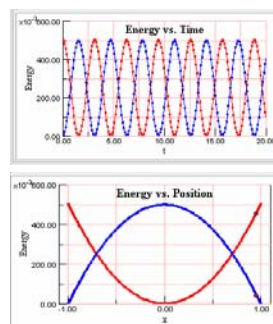
We have an object (mass m) attached to a massless spring. The object is on a horizontal frictionless surface. We move the object so the spring is stretched, and then we release it. The object oscillates back and forth.

To understand the motion, let's take a look at graphs of kinetic energy and elastic potential energy, first as a function of time and then as a function of position.

Which color goes with kinetic energy, and which with elastic potential energy?

[Simulation](#)

Energy Graphs, I



Which color goes with kinetic energy, and which with elastic potential energy?

[Simulation](#)

$$K = \frac{1}{2} mv^2$$

$$U = \frac{1}{2} kx^2$$

Energy graphs, I

Which graph is which?

1. The red one is the kinetic energy; the blue one is the potential energy.
2. The blue one is the kinetic energy; the red one is the potential energy.
3. The graphs are interchangeable so you can't tell which is which.

The spring is maximally stretched initially so U is the maximum at $t=0$. This is consistent with the red curve. On the other hand, when U is maximum, K is zero. This is consistent with the blue curve. Similarly, at $x=A$, U is the maximum and K is the minimum. This observation is also consistent with the red $E-x$ curve representing U and the blue $E-x$ curve representing K .

Energy graphs, II

How does the period of the energy- t graphs (T_E) compare to that of the $x-t$, $v-t$ or $a-t$ graphs (T)?

1. $T_E = T/2$
2. $T_E = T$
3. $T_E = 2T$

The period of the $K-t$ or $U-t$ graph is half that of the $x-t$ graph.

Reason: $K \propto \frac{1}{2} (1 - \cos(2\omega t))$ while $x \propto \cos(\omega t)$. So the period of the $K-t$ graph is π/ω while that of the $\omega-t$ graph is $2\pi/\omega$.

What determines the energy of a SHM?

$$E_{\text{tot}} = U(t) + K(t) = U_{\text{max}} = K_{\text{max}}$$

$$U_{\text{max}} = \frac{1}{2} kA^2 \quad (1)$$

$$K_{\text{max}} = \frac{1}{2} m v_{\text{max}}^2 \quad (2)$$

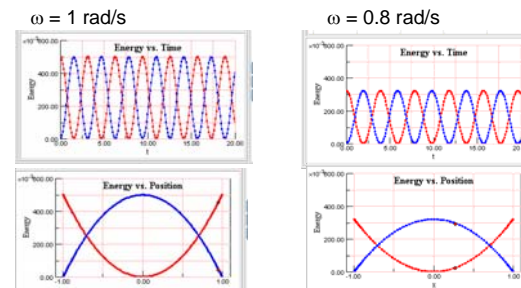
From the above, we can perceive the energy of a SHM as being either determined by k and A together (eqn. (1)) or m and v_{max} together (eqn. (2)) as follows:

By eqn. (1), if A is not changed, $k (= m\omega^2)$ determines the energy. That is, if a graph shows that A is not changed and m and/or ω is increased, E_{tot} is increased. By eqn. (2), if $v_{\text{max}} (= A\omega)$ is not changed, m determines the energy. That is, if a graph shows that v_{max} is not changed, increasing m increases E_{tot} .

A change in frequency, II

Simulation

The first set of graphs (upper: K and U vs. t ; lower: K and U vs. x) is for an angular frequency $\omega = 1$ rad/s. The second set of graphs is for $\omega = 0.8$ rad/s.



A change in frequency, II

The first set of graphs is for an angular frequency $\omega = 1$ rad/s. The second set of graphs is for $\omega = 0.8$ rad/s. Recall that $\omega = (k/m)^{1/2}$. So, either a reduction in k or an increase in m can produce the reduction in ω . Can you tell from the energy graphs whether it was k or m that had been changed in producing the reduction in ω ? You are given that the amplitude of oscillation has not been changed.

1. We decreased the spring constant
2. We increased the mass
3. We could have done one or the other, you can't tell the difference

A change in frequency, II

With the energy graphs, we can tell the difference. All the energy is in the spring initially, with the spring energy given by:

$$U_s = \frac{1}{2} kA^2$$

We are told that A is not changed and the graphs tell us that the energy stored in the spring for the second graph is smaller. Thus, we must have changed the spring constant, k .

On the other hand, if we had been told that v_{max} is not changed. You should instead consider $K_{\text{max}} = \frac{1}{2} m v_{\text{max}}^2$ and conclude from the graph that m is changed.

Thinking about time

An object attached to a spring is pulled a distance A from the equilibrium position and released from rest. It then experiences simple harmonic motion with a period T . The time taken to travel between the equilibrium position and a point A from equilibrium is $T/4$. How much time is taken to travel between points $A/2$ from equilibrium and A from equilibrium? Assume the points are on the same side of the equilibrium position, and that mechanical energy is conserved.

1. $T/8$
2. More than $T/8$
3. Less than $T/8$
4. It depends whether the object is moving toward or away from the equilibrium position

Thinking about time

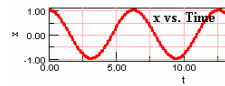
Let's say the object is A from equilibrium at $t = 0$, so the equation $x = A \cos(\omega t)$ applies.

Now just solve for the time t when the object is $A/2$ from equilibrium.

Thinking about time

Solve for t in the equation:

$$\frac{A}{2} = A \cos(\omega t) \Rightarrow \frac{1}{2} = \cos(\omega t)$$



Here we can use $\omega = \frac{2\pi}{T}$, so we need to solve:

$$\frac{1}{2} = \cos\left(\frac{2\pi t}{T}\right)$$

Take the inverse cosine of both sides. **We need to work in radians!**

$$\frac{\pi}{3} = \frac{2\pi t}{T} \Rightarrow t = \frac{T}{6}$$

This is more than $T/8$, because the object travels at a small average speed when it is far from equilibrium, where U is large and hence K is small.

A pendulum question

A simple pendulum is a ball on a string or light rod. We have two simple pendula of equal lengths. One has a heavy object attached to the string, and the other has a light object. Which has the longer period of oscillation?

1. The heavy one
2. The light one
3. Neither, they're equal

A pendulum question

Pull back the ball so it is a vertical distance h above the equilibrium position.

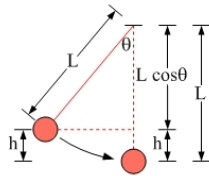
If you release the ball from rest, what is its speed when it passes through equilibrium?

Energy conservation: $mgh = \frac{1}{2}mv^2$

We get our familiar result $v = \sqrt{2gh}$

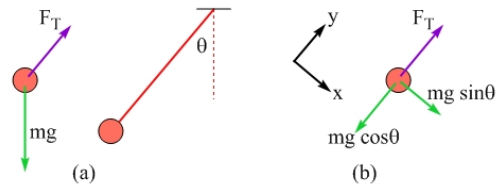
Does the ball's mass matter? No.

[Simulation](#)



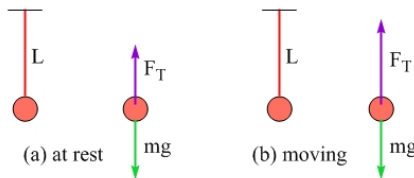
Free-body diagrams for a simple pendulum, I

Sketch a free body diagram for a pendulum when you release it from rest, after displacing it to the left.



Free-body diagrams for a simple pendulum, II

Sketch a free body diagram for the pendulum as it passes through equilibrium.



How should we analyze the pendulum? **Let's try torque.**

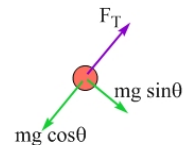
Angular Frequency of a pendulum

Take torques around the support point.

$$\Sigma \vec{\tau} = I\vec{\alpha}$$

$$-Lmg \sin \theta = mL^2 \alpha$$

$$\alpha = -\frac{g}{L} \sin \theta$$



For small angles we can say that $\sin \theta \approx \theta$

$$\alpha \approx -\frac{g}{L} \theta \quad \text{which has the SHM form} \quad \alpha = -\omega^2 \theta$$

So, the angular frequency is $\omega = \sqrt{\frac{g}{L}}$ [Simulation](#)

General features of simple harmonic motion

A system experiencing simple harmonic motion has:

- No loss of mechanical energy.
- A restoring force or torque that is proportional, and opposite in direction, to the displacement from equilibrium.

The motion is described by an equation of the form:

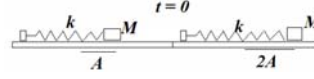
$$x = A \cos(\omega t)$$

where ω is the angular frequency of the system.

The period of oscillation is $T = \frac{1}{f} = \frac{2\pi}{\omega}$

Two Spring-block Systems

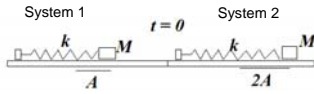
You have two identical springs and two identical blocks. You attach each block to a spring so you have two spring-block systems, and you set the blocks up to oscillate simultaneously on a frictionless horizontal surface. You pull the blocks so they stretch their respective springs, releasing them both from rest simultaneously. However, when you release the blocks one of them (we'll call this system 1) is displaced a distance A from equilibrium and the other (we'll call this system 2) is displaced a distance $2A$ from equilibrium.



(a) If the block in system 1 reaches a maximum speed v in its oscillations, what is the maximum speed reached by the block in system 2?

1. $v/2$ 2. v 3. $(2)^{1/2}v$ 4. $2v$ 5. $4v$

Two Spring-block Systems



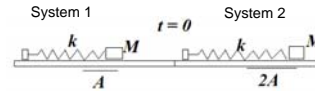
(b) If the block in system 1 experiences oscillations with a period T , what is the period of the oscillations experienced by the block in system 2?

1. $T/2$ 2. T 3. $(2)^{1/2}T$ 4. $2T$ 5. $4T$

(c) If the block in system 1 experiences oscillations with a period T , what is the period of the oscillations experienced by the block in system 2?

1. $F_{\max}/2$ 2. F_{\max} 3. $(2)^{1/2}F_{\max}$ 4. $2F_{\max}$ 5. $4F_{\max}$

Two Spring-block Systems



(d) If the potential energy stored in the spring in system 1 is U_1 when the block is first released from rest, what is the potential energy initially stored in the spring in system 2?

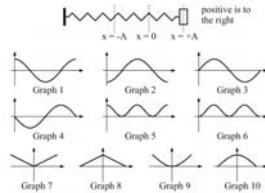
1. $U_1/2$ 2. U_1 3. $(2)^{1/2}U_1$ 4. $2U_1$ 5. $4U_1$

(e) At a particular instant, some time after being released, the block in system 1 is 20 cm from its equilibrium position. How far from equilibrium is the block in system 2 at that same instant?

1. 10 cm 2. 20 cm 3. 40 cm
4. There's not enough information to answer this question.

Example: Graphs for a Spring-block I

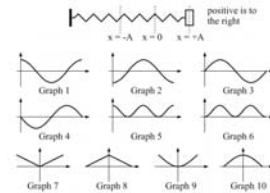
A block on a horizontal frictionless surface is attached to a spring. The spring is at its natural length when the block is at $x = 0$. At time $t = 0$, the block is released from rest at the point $x = A$. Graph 1 represents the position of the block as a function of time for one complete oscillation.



- (a) Which graph is the block's v-t graph?
 (b) Which graph is the block's a-t graph?
 (c) Which graph is the block's K-t graph?
 (d) Which graph is the block's U-x graph?
 (e) Which graph is the block's $|F|-x$ graph?

Example: Graphs for a Spring-block II

Continue from the last example. Consider a second experiment. Suppose instead of being released from rest, the block is given an initial push, so at $t = 0$ it starts moving from the origin to the right with speed v . Graph 3 represents the position of the block as a function of time for one complete oscillation. Considering one complete oscillation ...



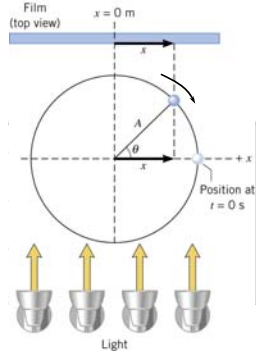
- (f) Which graph is the block's v-t graph?
 (g) Which graph is the block's a-t graph?
 (h) Which graph is the block's K-t graph?
 (i) Which graph is the block's U-x graph?
 (j) Which graph is the block's $|F|-x$ graph? (Neglect the initial push)

Connecting SHM with circular motion

Compare the motion of an object experiencing simple harmonic motion (SHM) to that of an object undergoing uniform circular motion.

[Simulation.](#)

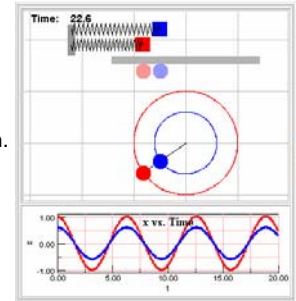
The time-dependent position $x(t)$ of the object on the spring is the same as that for the x-component of the circular motion. Both follow:
 $x = A \cos(\omega t)$



Amplitude does not affect frequency!

For simple harmonic motion, a neat feature is that the oscillation frequency is completely independent of the amplitude of the oscillation.

[Simulation.](#)

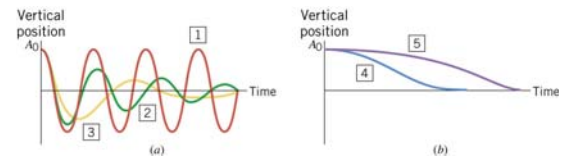


Damped Harmonic Motion

In simple harmonic motion, an object oscillated with a constant amplitude.

In reality, friction or some other energy dissipating mechanism is always present and the amplitude decreases as time passes.

This is referred to as **damped harmonic motion**.

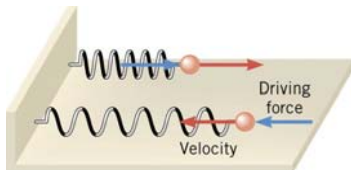


- 1) simple harmonic motion
- 2&3) underdamped
- 4) critically damped
- 5) overdamped

Driven Harmonic Motion

When a force is applied to an oscillating system at all times, the result is **driven harmonic motion**.

Here, the driving force has the same frequency as the spring system (i.e., $\sqrt{k/m}$) and always points in the direction of the object's velocity.



Resonance

Resonance is the condition in which a time dependent force can transmit large amounts of energy to an oscillating object, leading to a large amplitude motion.

Resonance occurs when the frequency of the force matches a natural frequency at which the object will oscillate.