Newton's Second Law for Rotation

Torque on a pulley

A constant force of F = 8 N is applied to a string wrapped around the outside of a pulley. The pulley is a solid disk of mass M = 2.0 kg and radius R = 0.50 m, and is mounted on a horizontal frictionless axle. What is the pulley's angular acceleration?

Simulation

Why are we told that the pulley is a solid disk?

So we know what to use for the rotational inertia. $I = \frac{1}{2}MR^2$

Torque on a pulley

Solution:

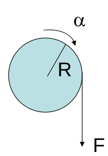
$\sum \vec{\tau} = I\vec{\alpha}$

$RF = \frac{1}{2}MR^2\alpha$

$F = \frac{1}{2}MR\alpha$

$\alpha = \frac{2F}{MR} = \frac{2 \times (8.0 \text{ N})}{(2.0 \text{ kg}) \times (0.5 \text{ m})} = 16 \text{ rad/s}^2$

$I = MR^2/2$

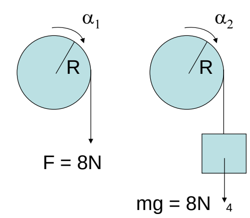


Two identical pulleys

Simulation

We take two identical pulleys, both with string wrapped around them. For the one on the left, we apply an 8 N force to the string. For the one on the right, we hang an object with a weight of 8 N. Which pulley has the larger angular acceleration? Why?

1. The one on the left
2. The one on the right
3. Neither, they're equal



Two identical pulleys

In both cases, the driving force is 8N. In the case on the left-hand-side (LHS), the entire 8N would be used to accelerate the rotation of the pulley. But in the case on the RHS, part of the 8N will be used to accelerate the block. So the pulley on the LHS will accelerate faster.

Two identical pulleys II

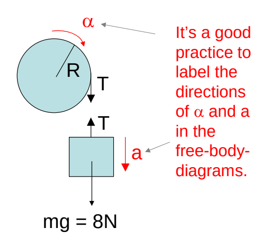
Suppose the pulleys have mass M_p = 2 kg, radius, R = 0.5 m. We already found that the angular acceleration of the pulley in the case on the LHS is 16 rad/s^2. Find that in the case on the RHS. Use g = 10 m/s^2.

Solution

$I = ½ M_pR^2 = ½ \times 2\text{kg} \times (0.5\text{m})^2$
$= 0.25 \text{ kgm}^2$

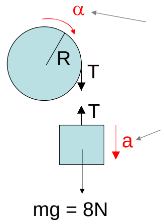
Applying the Newton's eqn. to the pulley, we get: $\tau = TR = I\alpha$

Applying the Newton's eqn. to the block, we get: 8N – T = ma



Two identical pulleys II

In the above two equations, there are three unknowns, namely T, α and a. With that, we need three equations to solve for their values. The third equation comes from the fact that the pulley and the block are connected by the string that wraps around the outer rim of the pulley. With that, $a = R\alpha$.



It's a good practice to label the directions of α and a in the free-body-diagrams.

Two identical pulleys II

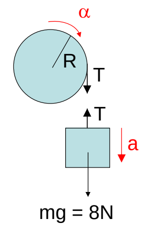
From the above discussions, we have the three equations:

$TR = I\alpha = ½ M_pR^2\alpha$ …………(1)
$8N – T = ma$ …………(2)
$a = R\alpha$ ………………(3)

Substitute (3) in (2),
$8N – T = mR\alpha$ ………………(4)

Eq. (1) $\Rightarrow T = ½ M_pR\alpha$ …………(1')

(4) + (1') $\Rightarrow 8N = (½ M_p+m)R\alpha$
$\Rightarrow \alpha = 8N / [(1/2 \times 2kg + 0.8kg) \times 0.5m)]$
$= 8.89$ rad/s^2.

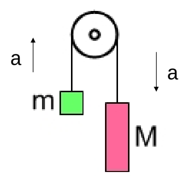


It's a good practice to label the directions of α and a in the free-body-diagrams.

This is less than that in case 1 as we predicted.

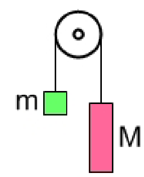
Atwood's machine

Atwood's machine involves one pulley, and two objects connected by a string that passes over the pulley. In general, the two objects have different masses.



Re-analyzing the Atwood's machine

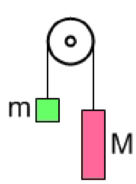
When we analyzed Atwood's machine in the past, we neglected the mass of the pulley (i.e., we assumed that the pulley is massless). If we include the mass of the pulley, we should expect the acceleration of the masses m and M to be:



1. larger than before.
2. smaller than before.
3. the same as before.

Re-analyzing the Atwood's machine

In an Atwood's machine, the driving force is (M – m)g. When the pulley is not massless, part of the driving force is diverted into accelerating the pulley's rotation.



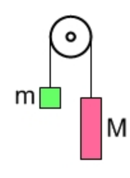
Acceleration in an Atwood's machine II

Problem: Find an expression for the acceleration of m and M in an Atwood's machine with a pulley mass of m_p.

Solution:

The pulley can be considered a solid disc, with $I = ½ m_pR^2$, where R is the radius of the pulley.

Physical picture: In the Atwood's machine, the tension force pulling on the heavier mass M is larger than that pulling on the lighter mass m. This results in a net clockwise torque and hence a clockwise angular acceleration in the pulley.



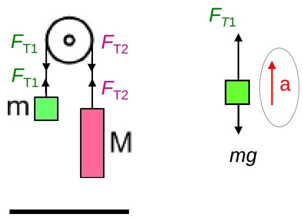
Acceleration in an Atwood's machine II

Step 1: Analyze the lighter block

Sketch the free-body diagram for the lighter block.

Choose a positive direction, and apply Newton's Second Law.

Let's choose positive to be up, i.e., in the direction of the acceleration.



$$\sum \vec{F} = m\vec{a}$$

$$+F_{T1} - mg = +ma$$

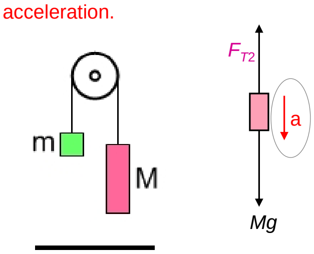
Acceleration in an Atwood's machine II

Step 2: Analyze the heavier block

Sketch a free-body diagram for the heavier object.

Choose a positive direction, and apply Newton's Second Law.

Choose positive down this time, to match the object's acceleration.



$$\sum \vec{F} = M\vec{a}$$

$$+Mg - F_{T2} = +Ma$$

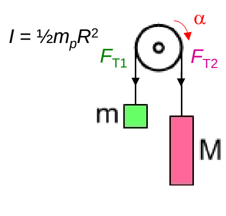
Acceleration in an Atwood's machine II

Step 3: Analyze the pulley

Sketch a free-body diagram for the pulley.

Choose a positive direction, and apply Newton's Second Law for rotation.

Choose positive clockwise, to match the pulley's angular acceleration.



$$\sum \vec{\tau} = I\vec{\alpha}$$

$$+RF_{T2} - RF_{T1} = +\left(\frac{1}{2}m_pR^2\right)\alpha$$

$$+F_{T2} - F_{T1} = +\left(\frac{1}{2}m_pR\right)\frac{a}{R}$$

equal

$$+F_{T2} - F_{T1} = +\frac{1}{2}m_pa$$

(Note that R cancels out so it is not needed to be specified.)

Acceleration in an Atwood's machine II

Step 4: combine the equations

Lighter block: $+F_{T1} - mg = +ma$

Heavier block: $+Mg - F_{T2} = +Ma$

Pulley: $+F_{T2} - F_{T1} = +\frac{1}{2}m_pa$

Add the equations: $+Mg - mg = +Ma + ma + \frac{1}{2}m_pa$

$$+Mg - mg = \left(M + m + \frac{1}{2}m_p\right)a$$

$$a = \frac{Mg - mg}{M + m + \frac{m_p}{2}}$$

Previous result for massless pulley

$$a = \frac{Mg - mg}{M + m}$$

The end