Torque and Rotational Inertia

Torque

Torque is the rotational equivalence of force. So, a net torque will cause an object to rotate with an angular acceleration.

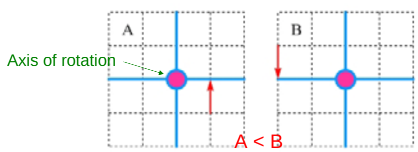
Because all rotational motions have an axis of rotation, <u>a torque must be defined about a rotational axis</u>.

A torque is a force applied to a point on an object about the axis of rotation.

The size of a torque depends on (1) <u>the size of the force applied</u> and (2) <u>its perpendicular distance from the axis of rotation</u> (which depends both on the direction of the force plus its physical distance from the axis of rotation).

A revolving door – effect of the physical distance from the axis of rotation

A force is applied to a revolving door that rotates about its center:

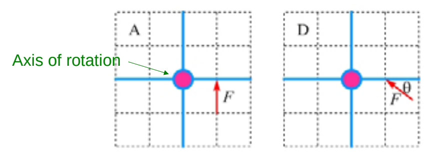


Rank these situations based on the magnitude of the torque experienced by the door, from largest to smallest.

1. B > A
2. B = A
3. B < A

A revolving door – effect of the direction of force

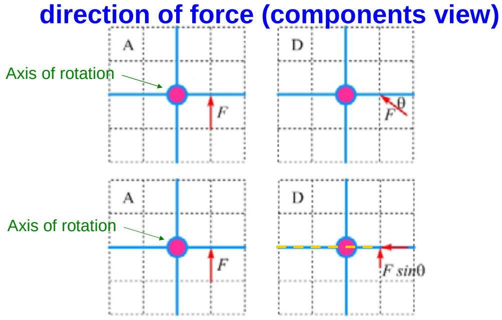
A force is applied to a revolving door that rotates about its center:



Rank the above two situations based on the magnitude of the torque experienced by the door, from largest to smallest.

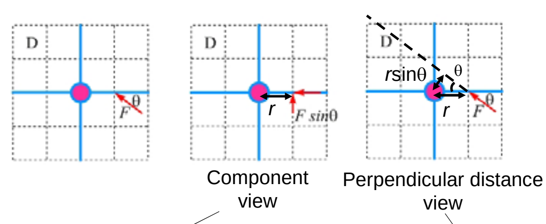
1. D > A
2. D = A
3. D < A

A revolving door – effect of the direction of force (components view)



The force component (Fcosθ) that acts along a line that passes through the axis of rotation does nothing.

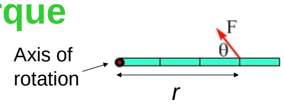
A revolving door – Components vs. perpendicular distance view



Perpendicular force *F*sinθ acting at a distance of *r* from the axis of rotation.

Force *F* acting at a perpendicular distance of *r*sinθ from the axis of rotation. Notice that *r*sinθ encompasses the effects of both the physical distance and direction of *F*.

Torque



In short, **torque is a vector** with magnitude given by:

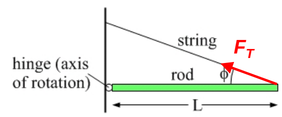
$$\tau = r\,F\sin\theta$$

where θ is the angle between ***r*** and ***F***.
Unit (SI): Nm

The direction of a torque (counterclockwise or clockwise) is determined by the direction of rotation the torque will cause an object to adopt from rest.

Example: Torque on a rod

Find the torque applied by the string on the rod shown below.

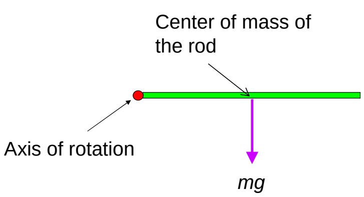


$\tau_{string} = LF_T\sin\phi$

Direction: anticlockwise

Torque due to the weight of an extended object

For an extended object (i.e., one whose mass is distributed over a volume in space), the torque due to its weight (*mg*) is that due to a force equal to *mg* acting downward at its center of mass.



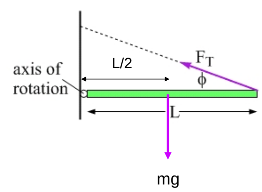
Net torque acting on the rod in Example 1

So, there are actually two torques acting on the rod about the hinge (labeled axis of rotation in the figure), one from the tension in the string, τ_{FT}, and one from the rod's own weight, τ_{mg}:

$\tau_{FT} = LF_T\sin\phi$

$\tau_{mg} = -(L/2)mg$

(Negative because it produces rotational motion in the opposite direction to τ_{FT}.)

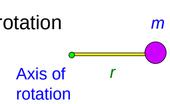


The **net torque** acting on the rod is a sum of the two torques:
$\tau = \tau_{FT} + \tau_{mg}$

Rotational Inertia or Moment of Inertia

The rotational equivalence of mass is moment of inertial, *I*. It accounts for how the mass of an extended object is distributed relative to the axis of rotation.

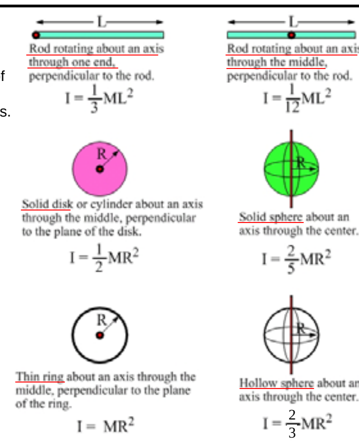
For a point mass *m* connected to the axis of rotation by a massless rod with length *r*, $I = mr^2$.



If the mass is distributed at different distances from the rotation axis, the moment of inertia can be hard to calculate. The expressions for *I* for several standard shapes are listed on the next page.

The distribution of the mass of a rod about an axis is more spread out when the axis is located at the edge of the rod than when it is located at the center of mass.

A table of rotational inertia

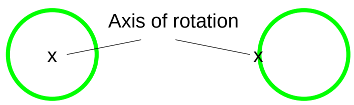


The parallel axis theorem

If you know the rotational inertia of an object of mass *m* when it rotates about an axis that passes through its center of mass, the object's rotational inertia when it rotates about a parallel axis a distance *h* away is:

$$I = I_{CM} + mh^2$$

Take a ring with radius *R* and mass *M* as an example:



$I = MR^2$ $I = MR^2 + MR^2 = 2MR^2$

Newton's Second Law for Rotation

The equation, $\sum \vec{\tau} = I\vec{\alpha}$

is the rotational equivalent of $\sum \vec{F} = m\vec{a}$.

Torque plays the role of force.

Rotational inertia plays the role of mass.

Angular acceleration plays the role of the acceleration.

Newton's First Law for Rotation

An object at rest tends to remain at rest, and an object that is spinning tends to spin with a constant angular velocity, **unless (1) it is acted on by a nonzero net torque or (2) there is a change in the way the object's mass is distributed**.

Based on $\Delta\omega/\Delta t = \alpha = \tau_{net} / I$, if either τ_{net} is nonzero or if I is changing with time, ω is changing with time.

Equilibrium

For an object to remain in equilibrium, two conditions must be met.

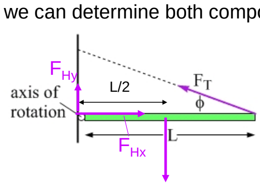
(1) The object must have no net force: $\sum \vec{F} = 0$

(2) and no net torque: $\sum \vec{\tau} = 0$ about any rotational axis.

Hinge Force

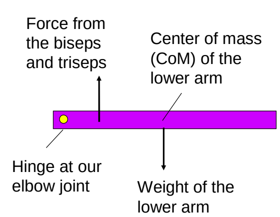
A hinge force (a vector), $\mathbf{F_H}$, generally exists at the hinge (usually the axis of rotation) of an hinged object at equilibrium

The figure below shows the hinge force (decomposed into x and y components) for the hinged rod discussed before. By appropriately using the requirements $\Sigma\mathbf{F} = 0$ and $\Sigma\tau = 0$, we can determine both components of F_H.



Example 1: Model of our lower arm

This is a model of our lower arm, with the elbow being the hinge.



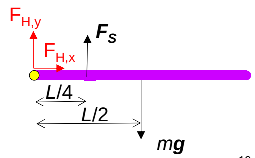
An equilibrium example

Example 1: Model of our lower arm

Draw a free-body diagram for a horizontal rod that is hinged at one end. The rod is held horizontal by an upward force applied by a spring scale ¼ of the way along the rod.

Find the reading on the scale (F_S) and the hinge force (F_H) in terms of mg, the weight of the rod if the rod is at equilibrium.

Let $\mathbf{F_H}$ be the hinge force, and we decompose it into $F_{H,x}$ and $F_{H,y}$ along the x and y direction, respectively.



Example 1: Model of our lower arm

To solve for F_S, we can use $\sum \vec{\tau} = 0$, calculated about **any** rotational axis, **EXCEPT** for the one that passes through the point where F_S is applied because this choice will make the torque coming from F_S go to zero (since *r* for that torque would be zero) and cause F_S to be eliminated from the equation.

Among the different possible choices for the rotational axis, we choose the one that passes through the hinge, with the advantage being that the unknown hinge force will get eliminated from the equation.

Let's define clockwise to be positive, and assume that F_S is upward and the rod has length *L*.

$$\sum \vec{\tau} = 0 \quad \Rightarrow \quad +\frac{L}{2} \times mg - \frac{L}{4} \times F_S = 0 \quad \Rightarrow F_S = 2mg$$

Example 1: Model of our lower arm

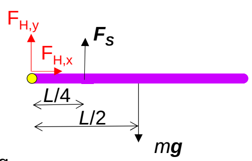
To find the hinge force, we can applied $\Sigma F_x = 0$ and $\Sigma F_y = 0$ to the system.

$\Sigma F_x = 0 \Rightarrow F_{Hx} = 0$

$\Sigma F_y = 0 \Rightarrow F_{Hy} + F_S - mg = 0$

$\Rightarrow F_{Hy} = mg - F_S$

$\Rightarrow F_{Hy} = mg - 2mg = -mg$



This negative sign means that the hinge force is actually pointing down, i.e., directed opposite to what is drawn for F_{Hy} in the picture.

Moving the spring scale

What, if anything, happens when the spring scale is moved farther away from the hinge? To maintain equilibrium:

1. The magnitude of the spring-scale force increases.
2. The magnitude of the spring-scale force decreases.
3. The magnitude of the downward hinge force increases.
4. The magnitude of the downward hinge force decreases.
5. Both 1 and 3
6. Both 1 and 4
7. Both 2 and 3
8. Both 2 and 4
9. None of the above.

Example 3 An A-shaped ladder

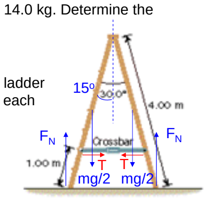
The drawing shows an A-shaped ladder. Both sides of the ladder are equal in length. This ladder is standing on a frictionless horizontal surface, and only the crossbar (which has a negligible mass) of the "A" keeps the ladder from collapsing. The ladder is uniform and has a mass of 14.0 kg. Determine the tension in the crossbar of the ladder.

Solution

Due to the symmetry of the ladder, the weight of the ladder can be taken to be acting equally at the mid-point of each side of the ladder.

In addition, due to the symmetry of the problem, it is sufficient to consider only one side of the ladder.

By $\Sigma F_y = 0$, $F_N = mg/2$



In writing the explicit terms for $\Sigma\tau = 0$, we choose the axis of rotation to be at the vertex of the ladder, perpendicular to the plane facing us.

$(2\text{ m})(mg/2)(\sin 15^o) + (3\text{ m})(T\cos 15^o) - (4\text{ m})(F_N \sin 15^o) = 0$

$\Rightarrow T = (4F_N \sin 15^o - mg\sin 15^o)/(3\cos 15^o)$

$\Rightarrow T = (2mg - mg)\tan 15^o/3 = (14\text{ kg})(9.8\text{ m/s2})\tan 15^o/3 = 12.3\text{ N}$

since $F_N = mg/2$